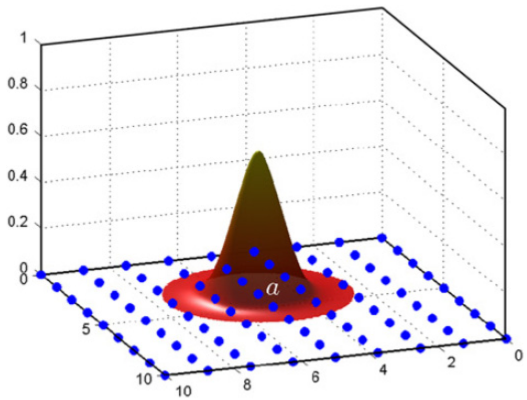
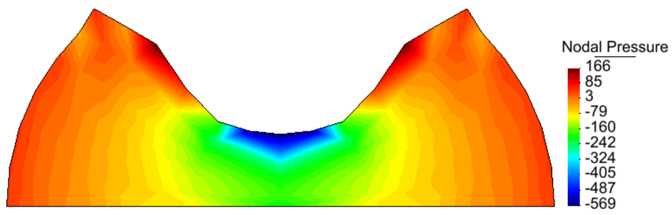
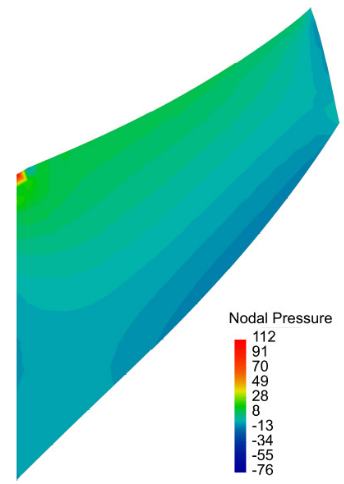
Meshfree Volume-Averaged Nodal Projection Method for Incompressible Media Problems







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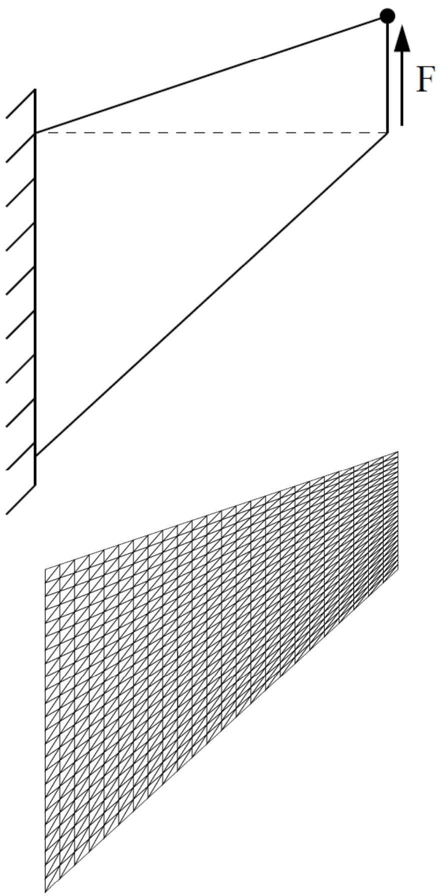
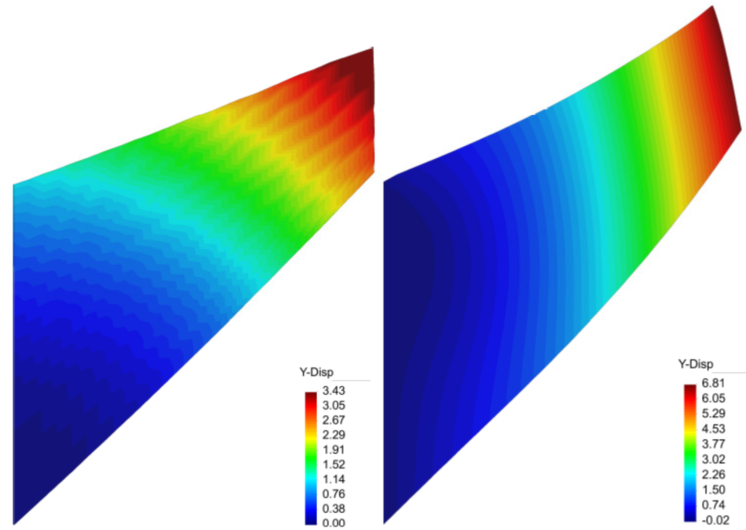
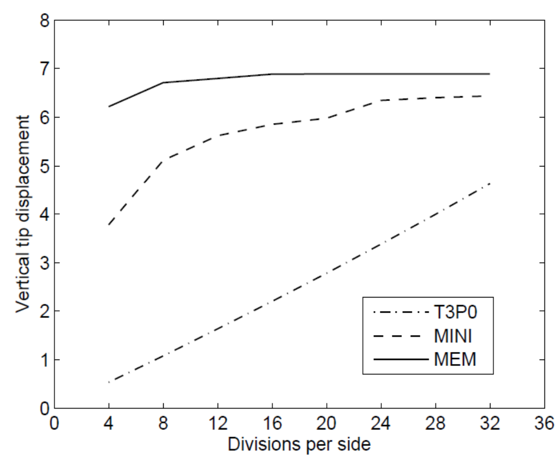
Mini-Symposium on Meshless and Related Methods
Barcelona ● Spain ● July 21, 2014

Motivation

Incompressible elasticity / 3-node triangular mesh / Locking







Motivation

- Some features of meshfree shape functions
  - ✓ Larger supports allow the triangle or tetrahedron to achieve larger deformations/distortions
  - ✓ Nodal information of a low-order triangular or tetrahedral mesh can be used to construct high-order approximations

The B-bar Approach (Linear Theory)

Modified potential energy functional **(Hughes, IJNME, 1980)**

$$\bar{\Pi}(\mathbf{u}) = \int_{\Omega} \Psi(\bar{\boldsymbol{\varepsilon}}(\mathbf{u}))\, d\Omega - \int_{\Omega} \mathbf{u} \cdot \mathbf{f}\, d\Omega - \int_{\Gamma_t} \mathbf{u} \cdot \mathbf{t}\, d\Gamma$$

where $\bar{\boldsymbol{\varepsilon}}(\mathbf{u})$ is a modified strain that precludes locking and leads to a displacement-based weak form:

$$\int_{\Omega} \delta\bar{\boldsymbol{\varepsilon}}(\mathbf{u}) : \bar{\boldsymbol{\sigma}}(\bar{\boldsymbol{\varepsilon}}(\mathbf{u}))\, d\Omega - \int_{\Omega} \delta\mathbf{u} \cdot \mathbf{f}\, d\Omega - \int_{\Gamma_t} \delta\mathbf{u} \cdot \mathbf{t}\, d\Gamma = 0$$

The B-bar Approach (Cont'd)

The usual split of the strain yields the modified strain:

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{u}) = \boldsymbol{\varepsilon}^{\mathrm{dev}}(\mathbf{u}) + \bar{\boldsymbol{\varepsilon}}^{\mathrm{dil}}(\mathbf{u}) \ ; \qquad \bar{\boldsymbol{\varepsilon}}^{\mathrm{dil}}(\mathbf{u}) = \tfrac{1}{3}\bar{\varepsilon}^{\mathrm{vol}}(\mathbf{u})\mathbf{I}$$

The modified stress becomes

$$\bar{\boldsymbol{\sigma}}(\bar{\boldsymbol{\varepsilon}}(\mathbf{u})) = \lambda\bar{\varepsilon}^{\mathrm{vol}}(\mathbf{u})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u})$$

So, the discrete equations (Ku = f) are obtained from

$$\int_\Omega \left(\delta\boldsymbol{\varepsilon}^{\mathrm{dev}} + \tfrac{1}{3}\delta\bar{\varepsilon}^{\mathrm{vol}}\mathbf{I}\right) : \left(\lambda\bar{\varepsilon}^{\mathrm{vol}}\mathbf{I} + 2\mu\boldsymbol{\varepsilon}\right)\, d\Omega - \int_\Omega \delta\mathbf{u}\cdot\mathbf{f}\, d\Omega - \int_{\Gamma_t} \delta\mathbf{u}\cdot\mathbf{t}\, d\Gamma = 0$$

So, everything reduces to compute $\bar{\varepsilon}^{\mathrm{vol}}$.

The F-bar Approach (Nonlinear Theory)

Modified potential energy functional **(Elguedj et al., CMAME, 2008)**

$$\bar{\Pi}(\boldsymbol{\chi}) = \int_{\Omega} \Psi(\bar{\mathbf{E}})\, d\Omega - \int_{\Omega} \mathbf{f}_0 \cdot \boldsymbol{\chi}\, d\Omega - \int_{\Gamma_t} \mathbf{t}_0 \cdot \boldsymbol{\chi}\, d\Gamma$$

with $\bar{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{C}} - \mathbf{I}) = \frac{1}{2}(\bar{\mathbf{F}}^{\mathrm{T}}\bar{\mathbf{F}} - \mathbf{I})$.

And $\bar{\mathbf{F}}$ is a modified deformation gradient that precludes locking and is computed using the usual split of the deformation gradient:

$$\bar{\mathbf{F}} = \bar{\mathbf{F}}^{\mathrm{dil}}\mathbf{F}^{\mathrm{dev}} = \left(\bar{J}^{1/3}\mathbf{I}\right)\left(J^{-1/3}\mathbf{F}\right) = \alpha\mathbf{F}$$

So, everything reduces to compute $\bar{J}$ in $\alpha = \left(\bar{J}/J\right)^{1/3}$.

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The F-bar Approach (Cont'd)

The first variation leads to the modified displacement-based weak form:

$$\int_{\Omega} \mathbf{S}(\bar{\mathbf{E}}) : \delta\bar{\mathbf{E}}\, d\Omega - \int_{\Omega} \mathbf{f}_0 \cdot \boldsymbol{\chi}\, d\Omega - \int_{\Gamma_t} \mathbf{t}_0 \cdot \boldsymbol{\chi}\, d\Gamma$$

The discrete equations (K_t u = -R) for the Newton's method are obtained from the second variation:

$$\int_{\Omega} \delta\bar{\mathbf{E}} : \bar{\mathcal{C}} : \mathrm{D}\bar{\mathbf{E}}[\mathbf{u}]\, d\Omega + \int_{\Omega} \mathbf{S}(\bar{\mathbf{E}}) : \mathrm{D}\delta\bar{\mathbf{E}}[\mathbf{u}]\, d\Omega$$
$$= -\left(\int_{\Omega} \mathbf{S}(\bar{\mathbf{E}}) : \delta\bar{\mathbf{E}}\, d\Omega - \int_{\Omega} \mathbf{f}_0 \cdot \boldsymbol{\chi}\, d\Omega - \int_{\Gamma_t} \mathbf{t}_0 \cdot \boldsymbol{\chi}\, d\Gamma \right)$$

where $\mathrm{D}(\cdot)[\mathbf{u}]$ denotes the directional derivative and $\bar{\mathcal{C}}$ is the modified elasticity tensor.

Volume-Averaged Nodal Projection (VANP) Method

- How to compute $\bar{\varepsilon}^{\text{vol}}$ and $\bar{J}$:

For $\bar{\varepsilon}^{\text{vol}}$ use the pressure constraint of the *u*-*p* form

$$\int_{\Omega} \delta p \left(\varepsilon^{\text{vol}} + \frac{p}{\lambda} \right) \, d\Omega = 0$$

Discretize the pressure parameter using meshfree shape functions:

$$p_h(\mathbf{x}) = \sum_b \phi_b(\mathbf{x}) p_b \; ; \;\; \delta p_h(\mathbf{x}) = \sum_c \phi_c(\mathbf{x}) \delta p_c$$

VANP Method (Cont'd)

The discrete pressure constraint after arbitrariness of δp_c is

$$\int_{\Omega} \phi_c \varepsilon^{\text{vol}} \, d\Omega + \left(\frac{1}{\lambda} \int_{\Omega} \sum_b \phi_c \phi_b \, d\Omega \right) p_b = 0$$

Solve for p_c by lumping the matrix using $\sum_b \phi_c \phi_b = \phi_b$:
(Ortiz et al., CMAME, 2010)

$$p_c = -\lambda \frac{\int_{\Omega_c} \phi_c \varepsilon^{\text{vol}} \, d\Omega}{\int_{\Omega_c} \phi_c \, d\Omega} = -\lambda \bar{\varepsilon}_c^{\text{vol}} = -\lambda \pi_c[\varepsilon^{\text{vol}}]$$

where the volume-averaged nodal projection (VANP) operator is

$$\pi_c[\cdot] \equiv \bar{\{\cdot\}} = \frac{\int_{\Omega_c} \phi_c \{\cdot\} \, d\Omega}{\int_{\Omega_c} \phi_c \, d\Omega}$$

VANP Method (Cont'd)

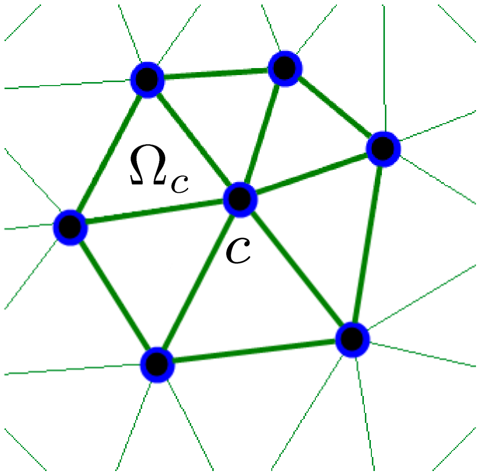
The operator can be used for $\bar{J}$ as well.

Finally, the discrete versions of $\bar{\varepsilon}^{\mathrm{vol}}$ and $\bar{J}$ are

$$\bar{\varepsilon}_h^{\mathrm{vol}} = \sum_c \phi_c(\mathbf{x})\pi_c[\varepsilon_h^{\mathrm{vol}}] = \sum_c \phi_c(\mathbf{x})\bar{\varepsilon}_c^{\mathrm{vol}}$$

$$\bar{J}_h = \sum_c \phi_c(\mathbf{x})\pi_c[J_h] = \sum_c \phi_c(\mathbf{x})\bar{J}_c$$

$$\pi_c[\cdot] \equiv \overline{\{\cdot\}} = \frac{\int_{\Omega_c} \phi_c \{\cdot\}\, d\Omega}{\int_{\Omega_c} \phi_c \, d\Omega}$$



(nodal domain)

LBB Stability

- Discretization in the (linear) VANP method:

$$\mathbf{u}_h = \sum_b \phi_b(\mathbf{x})\mathbf{u}_b$$

$$\bar{\varepsilon}_h^{\mathrm{vol}}(\mathbf{u}_h) = \sum_c \phi_c(\mathbf{x})\pi_c[\varepsilon_h^{\mathrm{vol}}(\mathbf{u}_h)]$$

- The second term comes from the pressure constraint of the *u*-*p* formulation

- ϕ_b and ϕ_c should be chosen to be inf-sup (LBB) stable

- Applies for the nonlinear case as well

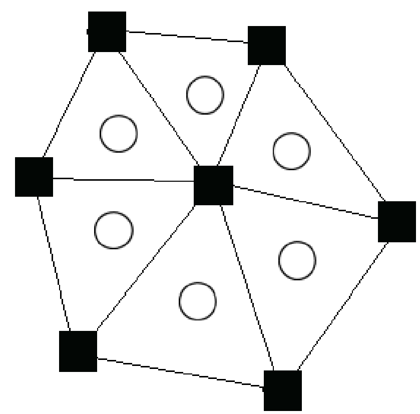
LBB Stability (Cont'd)

- Let's try to mimic well-known inf-sup stable FEs

MINI element **(Arnold et al., Calcolo, 1984)**

- ■ = globally continuous displacement and pressure DOFs
- ○ = cubic bubble (N4 = N1*N2*N3) displacement DOFs
- LBB stable

- Linear displacement w/bubble enrichment
- Linear Pressure



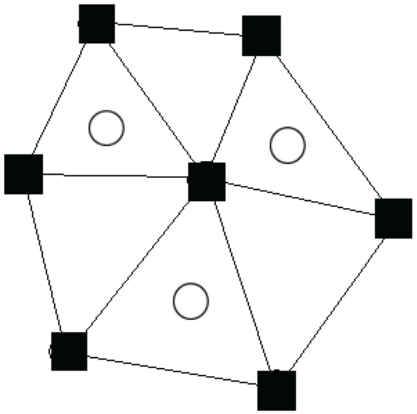
(used in Ortiz et al., CMAME, 2010)

LBB Stability (Cont'd)

MINI* element **(Kim & Lee, ACM, 2000)**

- Displacement and pressure DOFs (~ MINI element)
- Bubble node every other element
- LBB stable

- Linear displacement w/bubble enrichment
- Linear Pressure

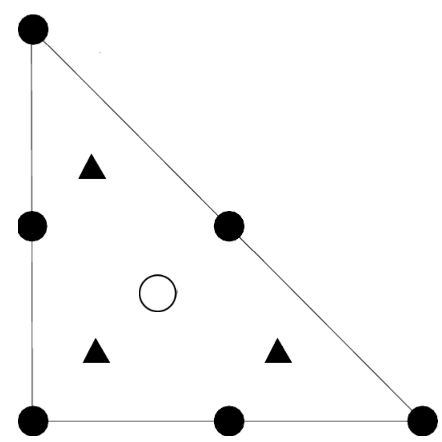


LBB Stability (Cont'd)

Conforming Crouzeix & Raviart element **(1974)**

- ● = globally continuous displacement DOFs
- ○ = cubic bubble (N4 = N1*N2*N3) displacement DOFs
- ▲ = globally discontinuous pressure DOFS
- LBB stable

- Quadratic displacement w/bubble enrichment
- Linear Pressure

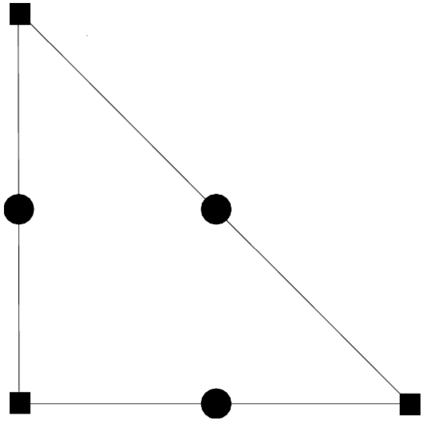


LBB Stability (Cont'd)

Taylor-Hood element **(1974)**

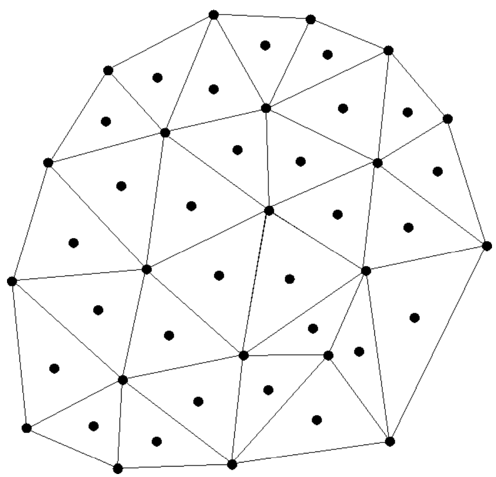
- ■ = globally continuous displacement and pressure DOFs
- ● = globally continuous displacement DOFs
- LBB stable

- Quadratic displacement
- Linear Pressure



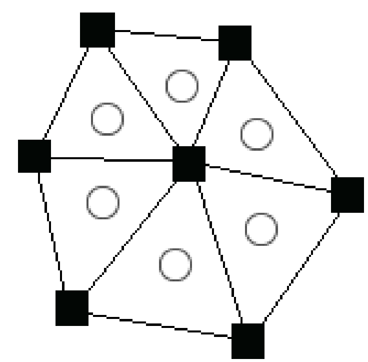
LBB Stability (Cont’d)

- Let’s mimic the MINI element



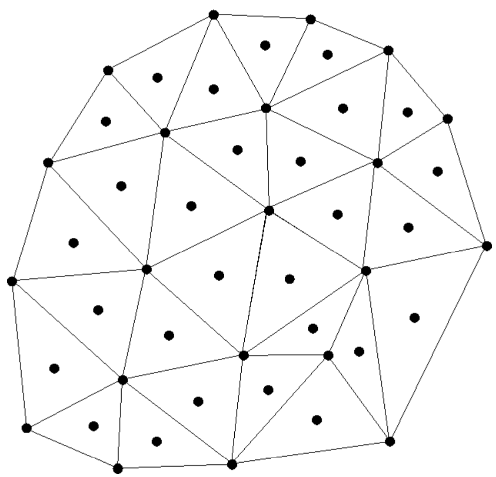
Vertex nodes	$\bar{\varepsilon}_h^{\text{vol}}(\mathbf{u}_h)$	}	First-order meshfree shape functions
Vertex + interior nodes	$\mathbf{u}_h$	}	First-order meshfree shape functions

$\approx$



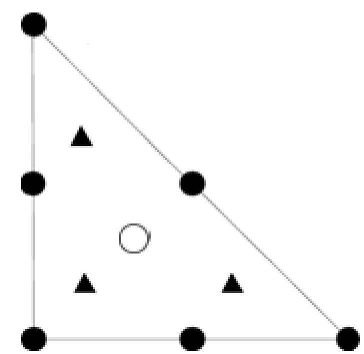
LBB Stability (Cont'd)

- Let's mimic the conforming Crouzeix & Raviart element



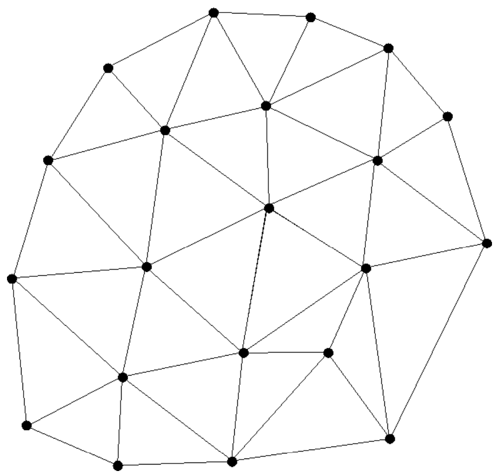
Vertex nodes	$\bar{\varepsilon}_h^{\text{vol}}(\mathbf{u}_h)$	}	First-order meshfree shape functions
Vertex + interior nodes	$\mathbf{u}_h$	}	Second-order meshfree shape functions

$\approx$



LBB Stability (Cont'd)

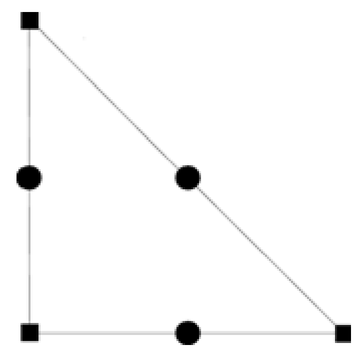
- Let's mimic the Taylor-Hood element



Vertex nodes $\bar{\varepsilon}_h^{\mathrm{vol}}(\mathbf{u}_h)$ } First-order meshfree shape functions

Vertex nodes $\mathbf{u}_h$ } Second-order meshfree shape functions

$\approx$



Numerical Examples: Notation

$$\text{VANP-}T_p^a/T_q\text{-b}$$

VANP = volume averaged nodal projection method

T = simplicial tessellation

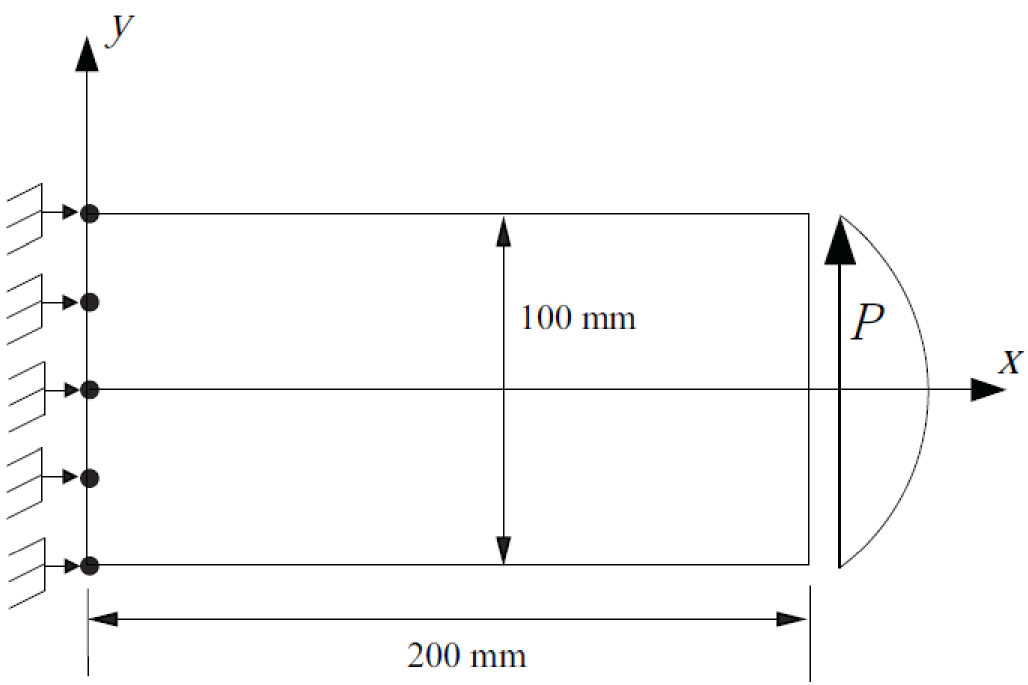
p = order of the displacement field

q = order of the projected strain

a = '+' (full bubble enrichment) or '*' (a bubble every other cell)

b = '*m*' (maxent) or '*r*' (rpim)

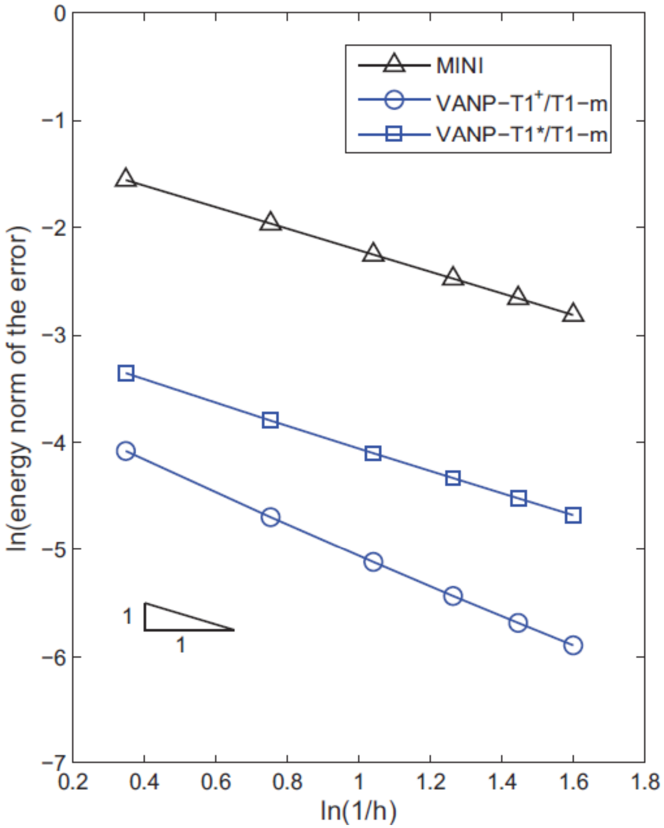
Numerical Example: Cantilever Beam

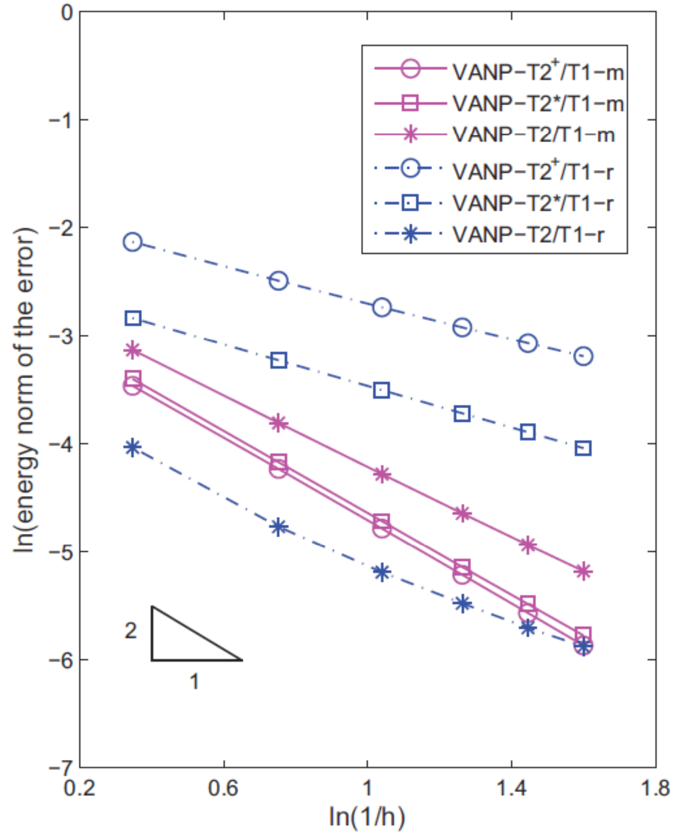


- P = -5000 N
- E = 210000 Mpa
- ν = 0.4999

(Timoshenko & Goodier, 1970)

Cantilever Beam (Cont'd)

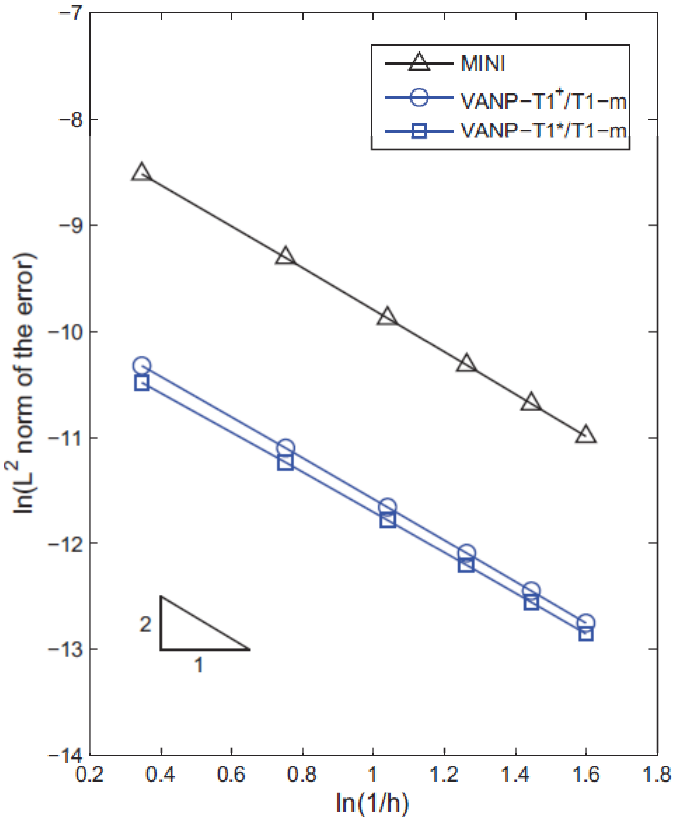


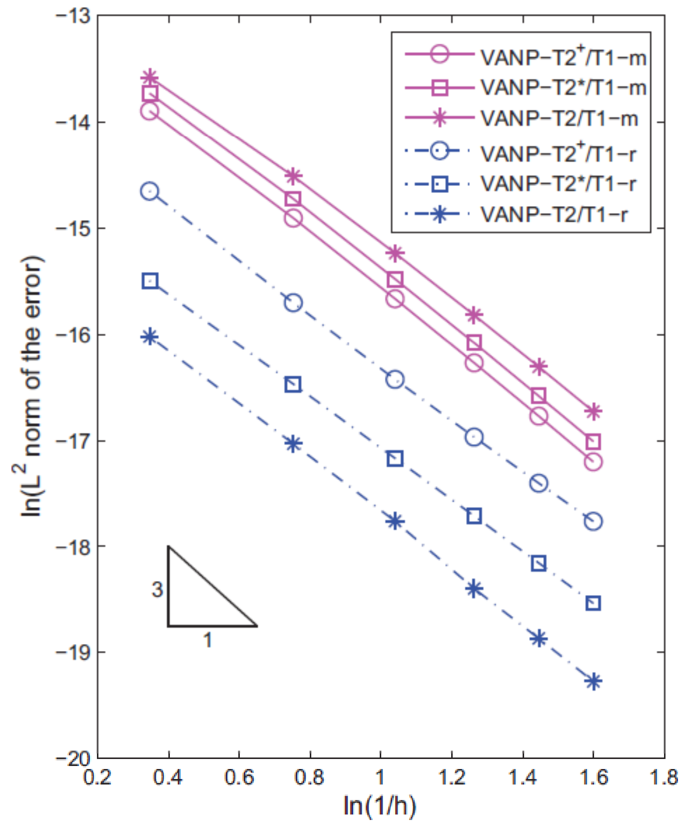


First-order approximations

Second-order approximations

Cantilever Beam (Cont'd)

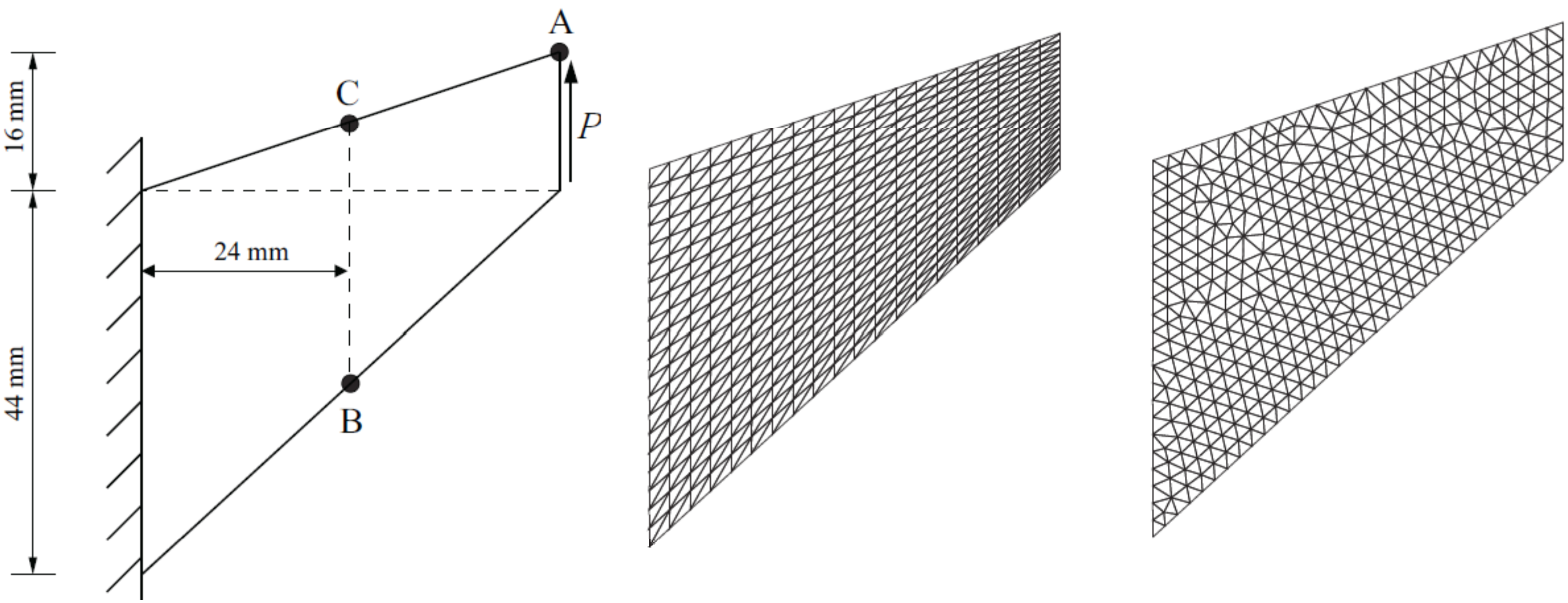




First-order approximations

Second-order approximations

Numerical Example: Cook's Membrane

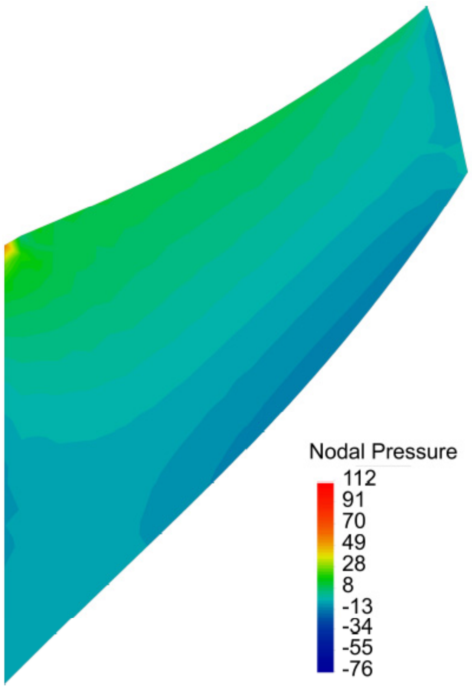


- P = 100 N
- E = 240.565 MPa ; ν = 0.4999

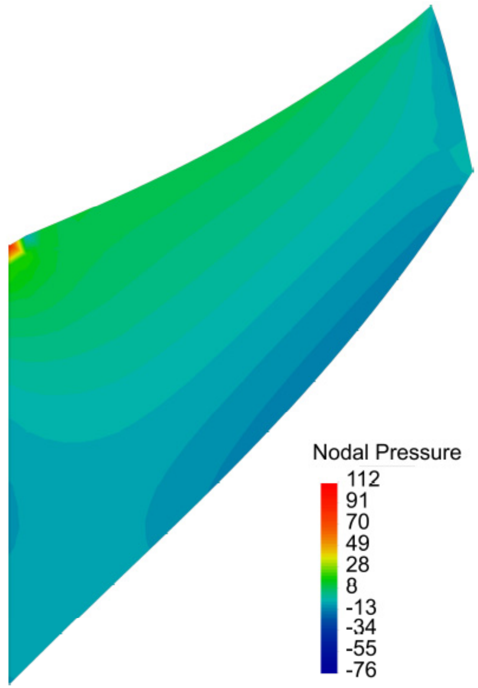
(Elguedj et al., CMAME, 2008)

Cook's Membrane (Cont'd)

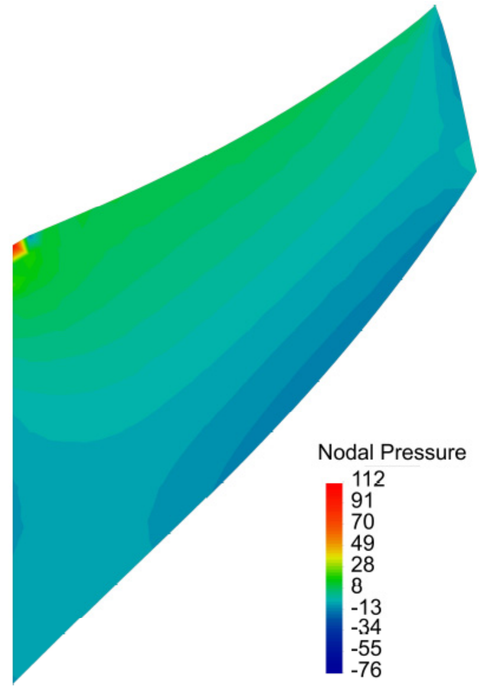
First-order approximations (unstructured tessellation)



MINI



VANP-T_1^+/T_1-m



VANP-T_1^*/T_1-m

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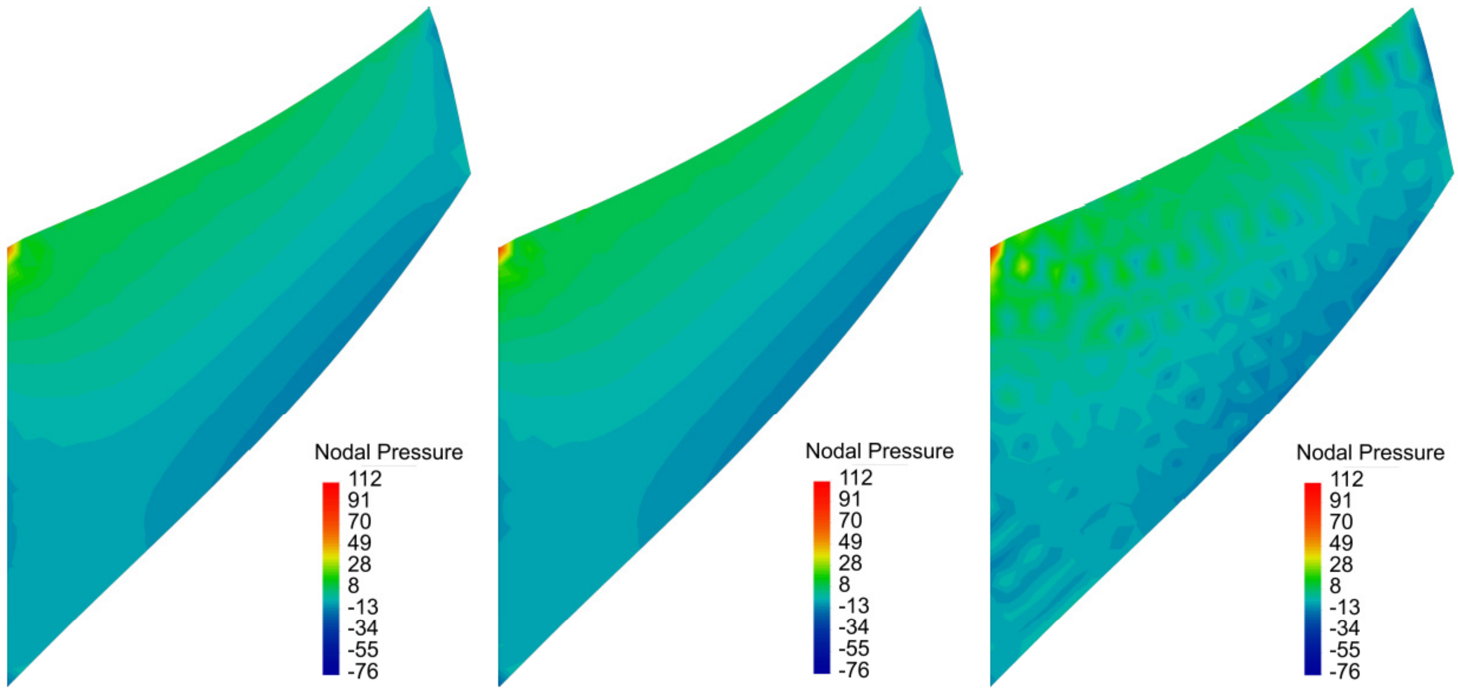
Cook’s Membrane (Cont’d)

Second-order approximations (unstructured tessellation)



VANP-T_2^+/T_1-r   VANP-T_2^*/T_1-r   VANP-T_2/T_1-r

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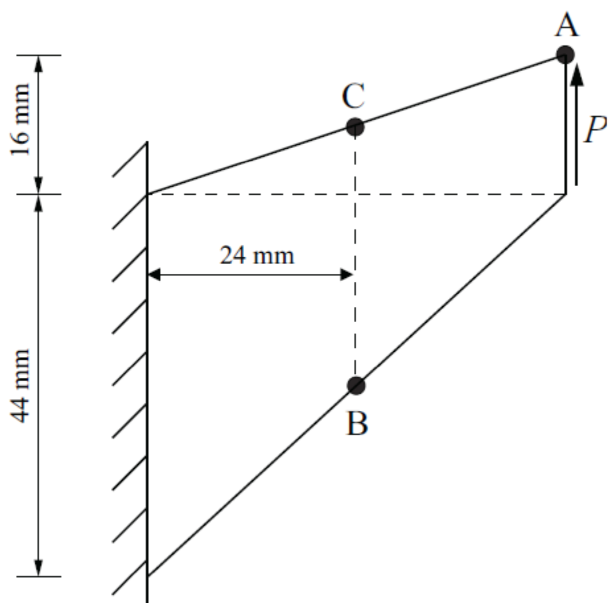
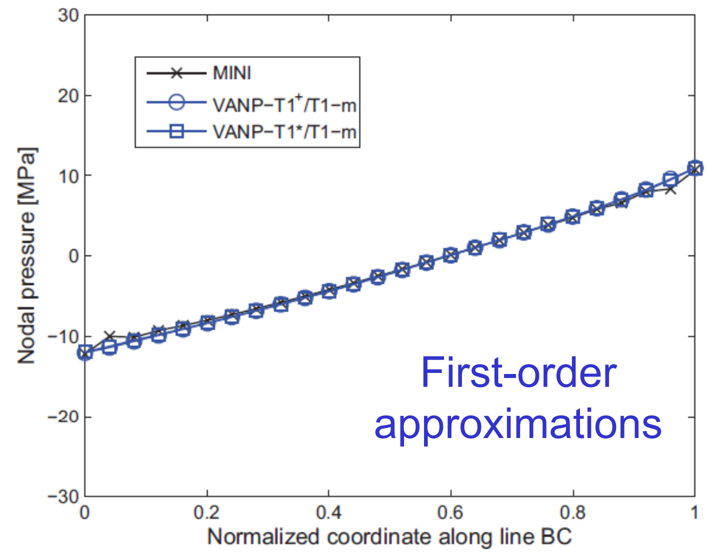
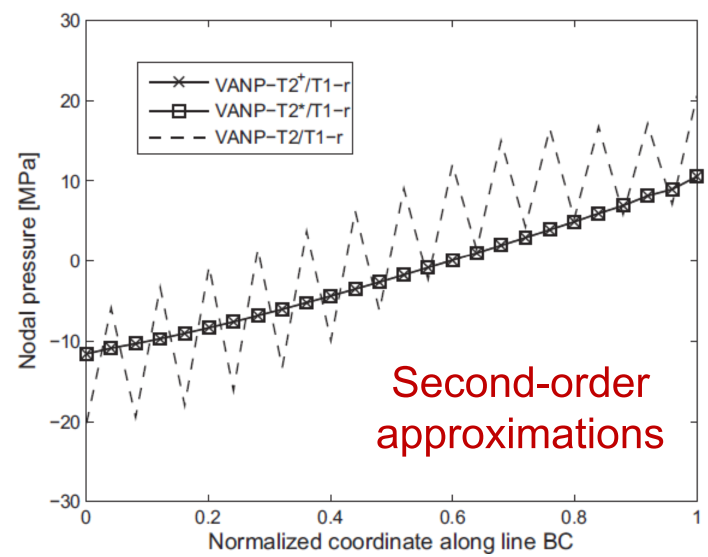
Cook's Membrane (Cont'd)







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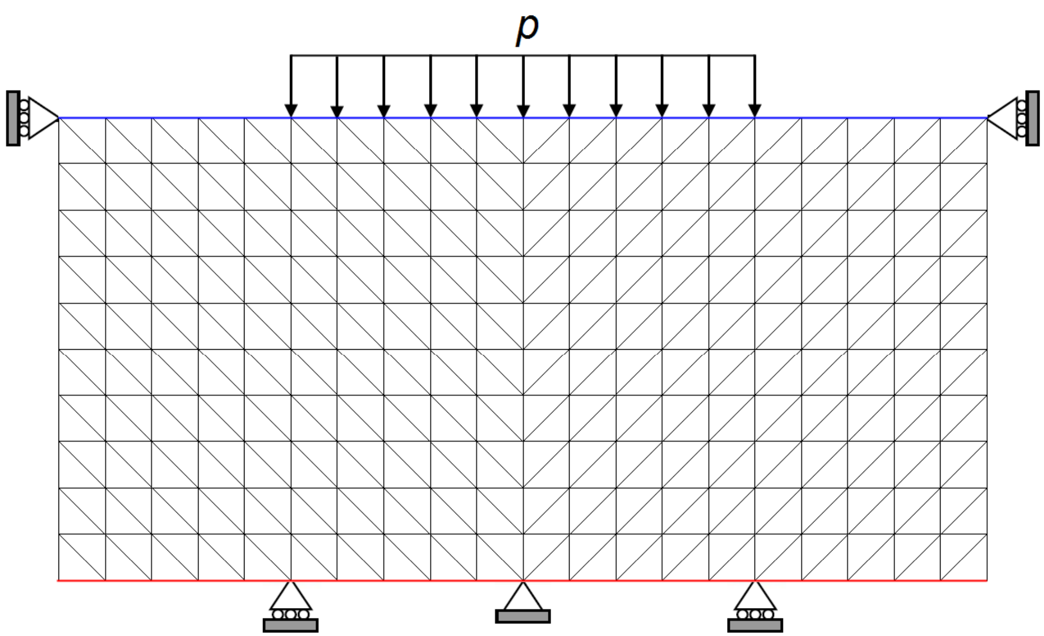
Numerical Example: 2D Rubber Block Compression



- p = 500 MPa
- neo-Hookean material: μ = 80.194 MPa ; κ = 400889.806 MPa

(Elguedj et al., CMAME, 2008)

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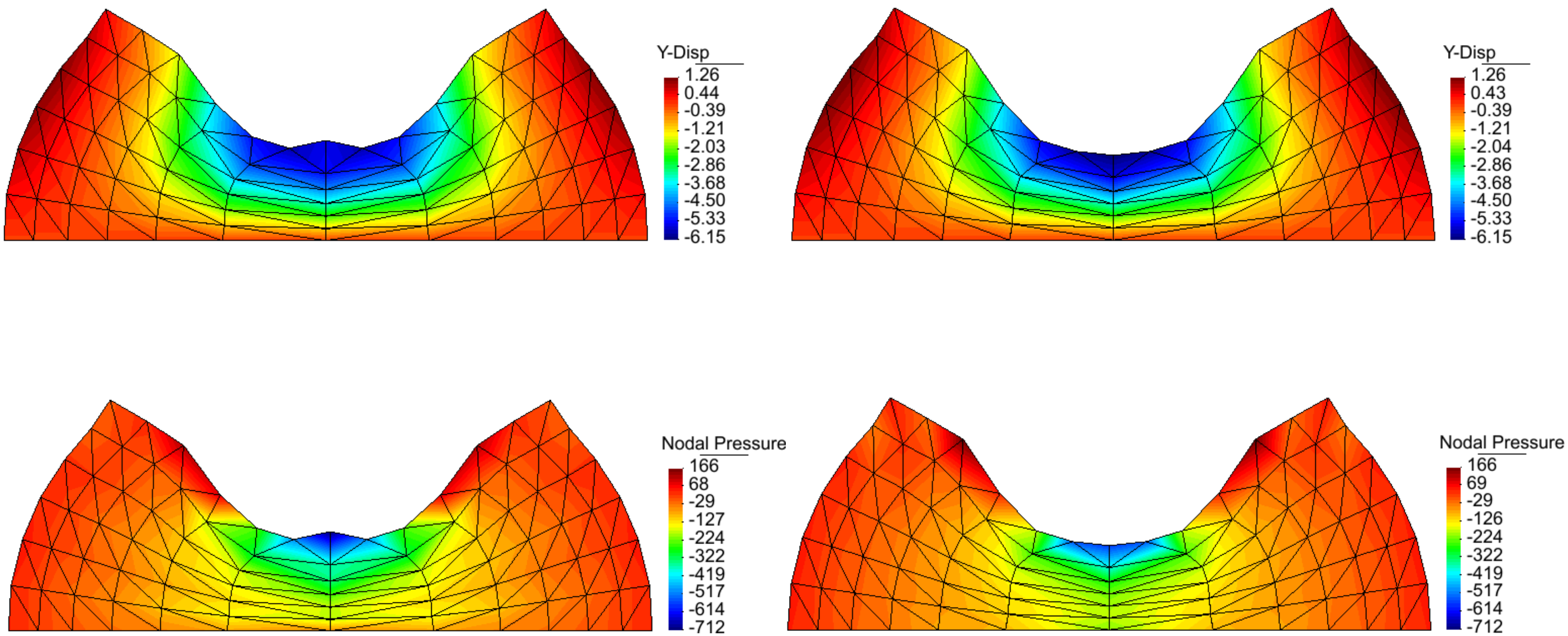
2D Rubber Block Compression (Cont'd)



MINI

VANP-T_1^+/T_1-m

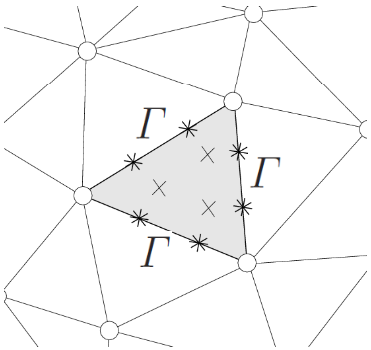
Numerical Integration

- Integration errors in Galerkin meshfree methods

  (Dolbow & Belytschko, CM, 1999; Babuska et al., IJNME, 2008)

- Cell smoothing integration schemes the best available

  (Chen et al., IJNME, 2001 & 2013; Duan et al., IJNME, 2012 & 2014)



- VANP uses the method of Duan et al. (2014)

  Presentation by **Qinglin Duan** forthcoming in the afternoon sessions:

  “Efficient and highly accurate high order meshless methods based on Hu-Washizu variational principle”

Summary

The VANP approach for incompressible media problems

- uses low-order simplicial meshes regardless of the order of the approximation
- needs bubble enrichment to obtain smooth pressure fields
- mimics reasonably well the MINI, MINI* and conforming Crouzeix & Raviart LBB stable finite elements
- converges optimally in the energy- and L2-norms when maximum-entropy shape functions are used
- achieves larger deformations than the MINI element