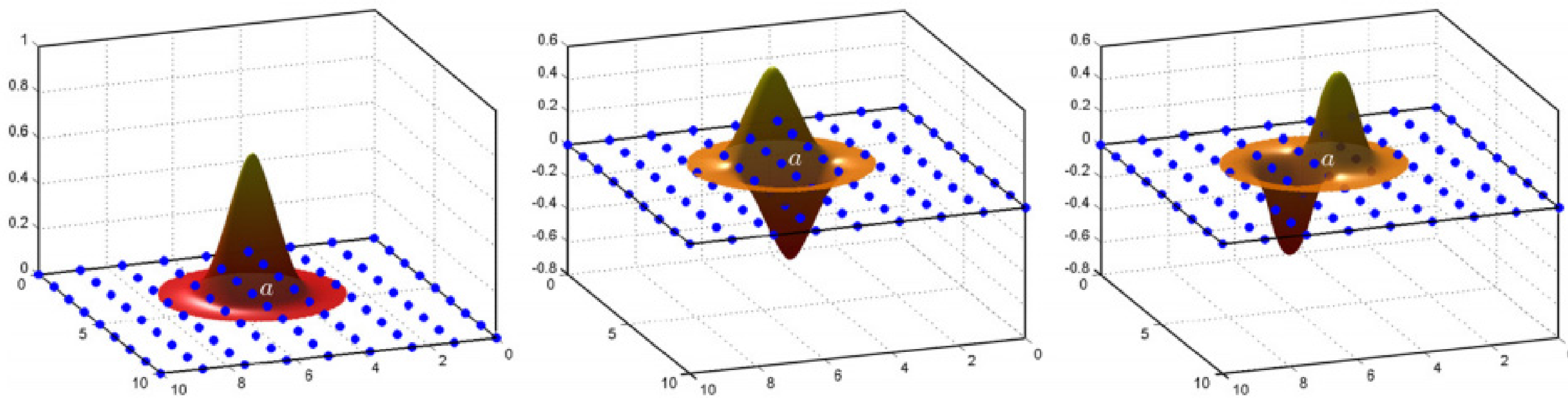
Universidad de Chile

Consistent and Stable Meshfree Galerkin Methods Using The Virtual Element Decomposition



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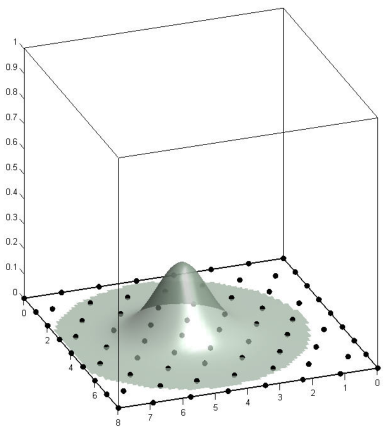
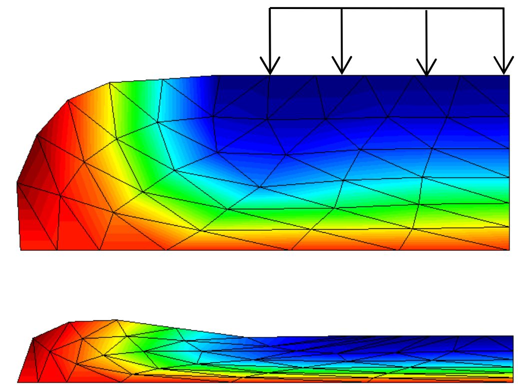
Meshfree basis functions



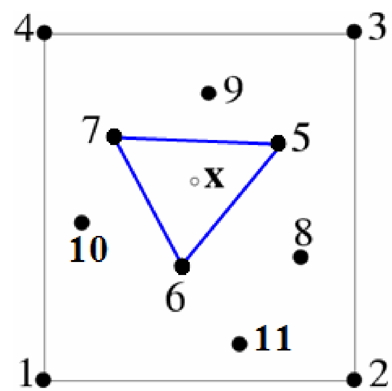
- Rational (non-polynomial)
- At least linearly precise
- Smooth
- $\phi_a(\boldsymbol{x}_b) \neq \delta_{ab}$
- Large but compact support
- Based on nodes
- Useful for large deformations



Meshfree approximation:

$$\boldsymbol{u}^h(\boldsymbol{x}) = \sum_{a=1}^{m} \phi_a(\boldsymbol{x})\boldsymbol{u}_a$$

Linear polynomial + additional non-polynomial part



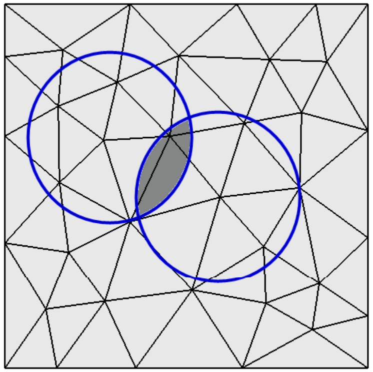
$m > N$

Motivation (Cont'd)

Stiffness matrix (Galerkin weak form)

$$\mathsf{K}_{ab} = \int_{\Omega} \mathsf{B}_a^{\mathsf{T}} \mathsf{C} \mathsf{B}_b \, dx \qquad \mathsf{B}_a = \begin{bmatrix} \phi_{a,x} & 0 \\ 0 & \phi_{a,y} \\ \phi_{a,y} & \phi_{a,x} \end{bmatrix}$$

Integration cells



- Integration errors leads to
  - failure to meet consistency (patch test is not satisfied)
  - stability issues (rank deficient stiffness matrix)
- Existing integration methods only recover consistency

  [Ortiz et al. (2010), Duan et al. (2012, 2014), Chen et al. (2013)]

New method: recovers consistency and ensures stability by means of the virtual element decomposition

Meshfree Method - Virtual Element Decomposition

Following VEM literature: Beirão da Veiga et al. (2013), Gain et al. (2014)

- Bilinear form: $a_E(\boldsymbol{u}, \boldsymbol{v}) = \int_E \boldsymbol{\sigma}(\boldsymbol{u}) : \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}$
- $\mathcal{W}$ the space containing linear polynomials + some additional non-polynomials. $\mathcal{P}$ the space of linear polynomials.
- $\mathit{\Pi}$ a projection operator onto $\mathcal{P}$ and is required to satisfy

$$a_E(\boldsymbol{p}, \boldsymbol{v} - \mathit{\Pi}\boldsymbol{v}) = 0 \quad \forall \boldsymbol{p} \in \mathcal{P}, \; \boldsymbol{v} \in \mathcal{W}$$

- The meshfree approximation $\boldsymbol{v}^h(\boldsymbol{x}) = \sum_{a=1}^{m} \phi_a(\boldsymbol{x}) \boldsymbol{v}_a \in \mathcal{W}$

Virtual element decomposition: $\boldsymbol{v}^h = \mathit{\Pi}\boldsymbol{v}^h + (\boldsymbol{v}^h - \mathit{\Pi}\boldsymbol{v}^h)$

$$a_E(\boldsymbol{v}^h, \boldsymbol{v}^h) = \underbrace{a_E(\mathit{\Pi}\boldsymbol{v}^h, \mathit{\Pi}\boldsymbol{v}^h)}_{\text{consistency}} + \underbrace{a_E(\boldsymbol{v}^h - \mathit{\Pi}\boldsymbol{v}^h, \boldsymbol{v}^h - \mathit{\Pi}\boldsymbol{v}^h)}_{\text{stability}}$$

Meshfree Method - Virtual Element Decomp. (Cont'd)

The VE decomposition gives the usual VEM stiffness matrices

$$a_E(\boldsymbol{v}^h, \boldsymbol{v}^h) = a_E(\Pi \boldsymbol{v}^h, \Pi \boldsymbol{v}^h) + a_E(\boldsymbol{v}^h - \Pi \boldsymbol{v}^h, \boldsymbol{v}^h - \Pi \boldsymbol{v}^h)$$
$$= \mathbf{q}^{\mathrm{T}} \left(\mathbf{K}_E^{\mathrm{consistency}} + \mathbf{K}_E^{\mathrm{stability}} \right) \mathbf{d}$$

In computing the stiffness, this approach differs to VEM in

$$q_{ia} = \frac{1}{2|E|} \int_{\partial E} \phi_a n_{iE} \, \mathrm{d}s, \quad i = 1, 2$$ ... is not exact

... contributions from outside the cell

$$\overline{\phi}_a = \frac{1}{N} \sum_{J=1}^{N} \phi_a(\boldsymbol{x}_J) \neq \frac{1}{N}$$ since $\phi_a(\boldsymbol{x}_b) \neq \delta_{ab}$

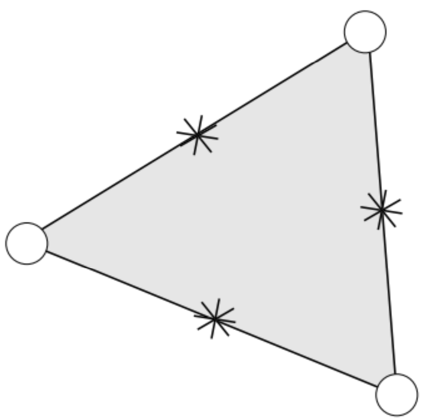
[Ortiz-Bernardin, Russo & Sukumar, IJNME, 2017]

Numerical Examples

Acronyms:

MEM = standard meshfree method (volume integrals)
MEM-VED = meshfree method – virtual element decomp.

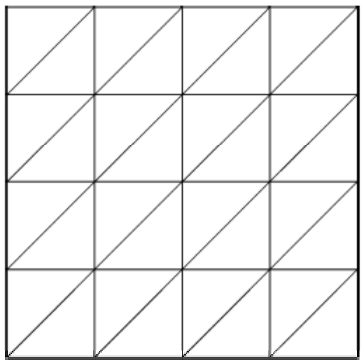
Consistent and stable integration rule in the MEM-VED:



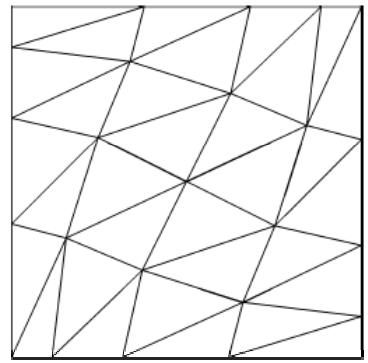
- 1-pt per face (also for tetrahedra)
- no need for derivatives

Example: Displacement Patch Test

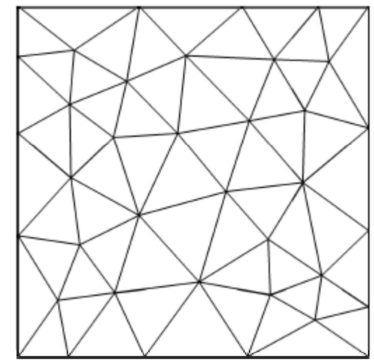
Regular



Distorted



Unstructured



A linear displacement field is prescribed on the entire boundary

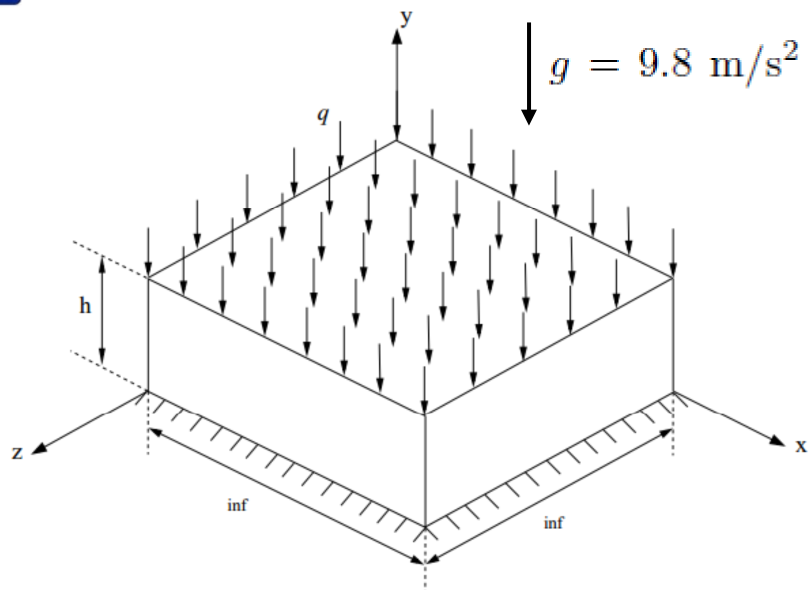
Relative error in the H^1 seminorm

Method	Gauss rule	Regular	Distorted	Unstructured
MEM	1-point	2.2×10^{-3}	2.1×10^{-1}	2.7×10^{-1}
MEM	3-point	9.7×10^{-4}	1.9×10^{-2}	4.2×10^{-2}
MEM	6-point	8.4×10^{-4}	6.5×10^{-3}	1.5×10^{-2}
MEM	12-point	1.5×10^{-4}	3.4×10^{-3}	7.7×10^{-3}
MEM–VED	1-point	9.2×10^{-14}	5.2×10^{-13}	1.0×10^{-12}

Relative error in the L^2 norm

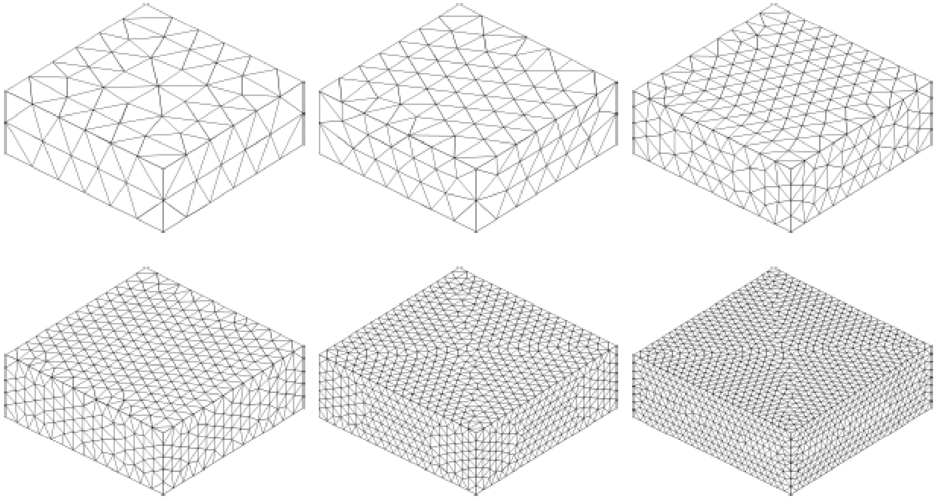
Method	Gauss rule	Regular	Distorted	Unstructured
MEM	1-point	4.7×10^{-4}	3.0×10^{-2}	2.6×10^{-2}
MEM	3-point	1.8×10^{-4}	2.8×10^{-3}	3.8×10^{-3}
MEM	6-point	1.9×10^{-4}	1.2×10^{-3}	1.4×10^{-3}
MEM	12-point	3.2×10^{-5}	5.8×10^{-4}	7.2×10^{-4}
MEM–VED	1-point	8.7×10^{-15}	1.7×10^{-13}	2.5×10^{-13}

Example: Infinite Elastic Stratum

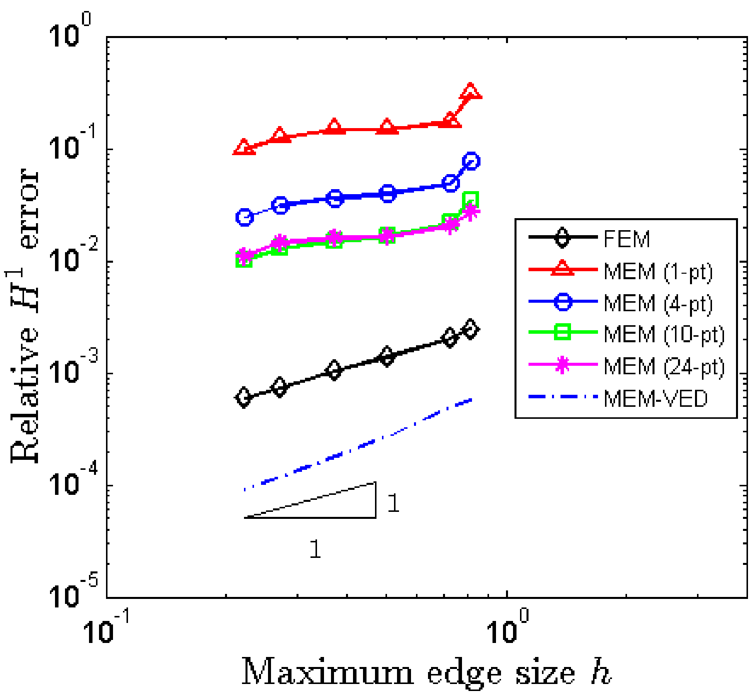


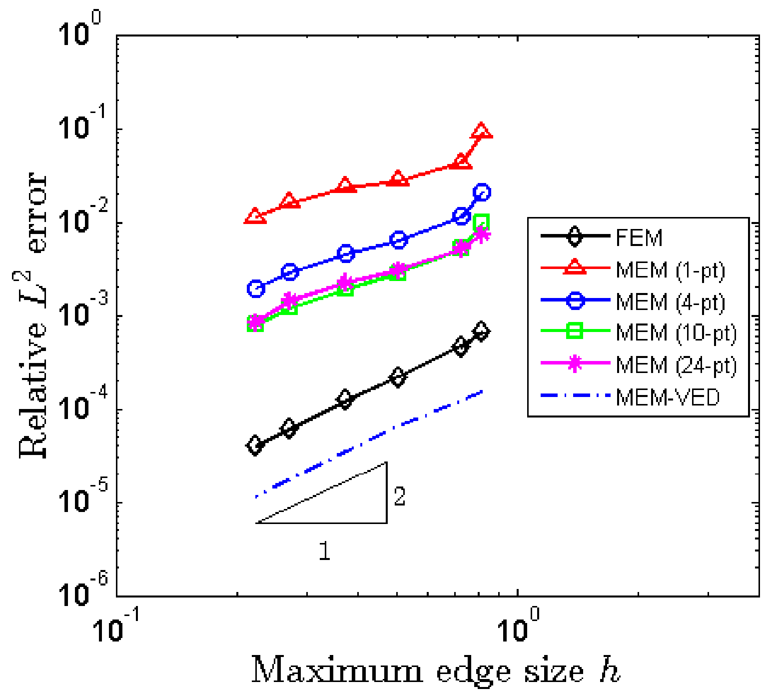
- $q = 10^6$ Pa
- $E = 4\mathrm{x}10^7$ Pa
- $\nu = 0.3$
- h = 1 m
- $\rho = 1900$ kg/m2
- $u_z = u_x = 0$

(Exact solution taken from Duan et al., CMAME, 2014)



Infinite Elastic Stratum (Cont'd)





Convergence rates

Possible Extensions

- Higher-order meshfree approximations
- Use of more general integration cells
- Meshfree nodal integration methods
- Large deformations