THE VIRTUAL ELEMENT DECOMPOSITION: A NEW PARADIGM FOR DEVELOPING NODAL INTEGRATION SCHEMES FOR MESHFREE GALERKIN METHODS

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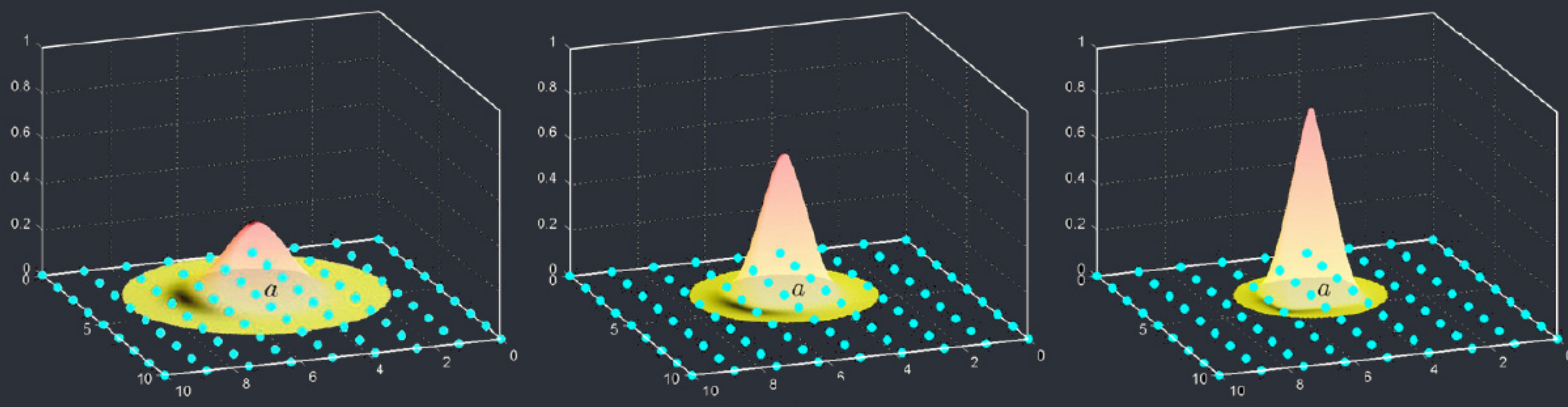

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Funded through grant CONICYT/FONDECYT No. 1181192

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WONAPDE 2019

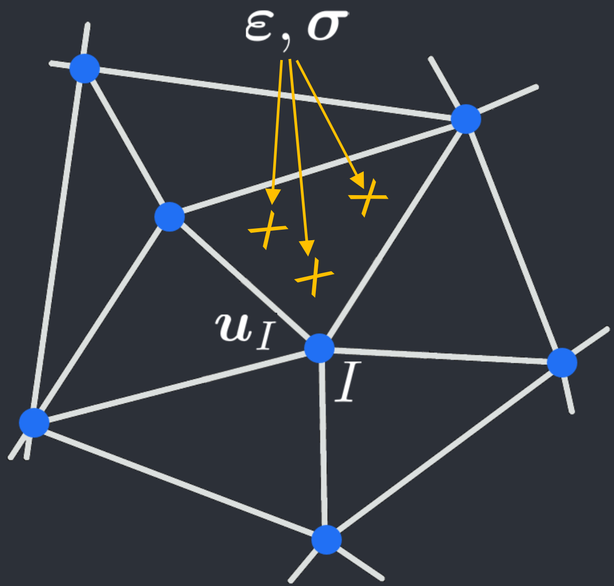
Sixth Chilean Workshop on Numerical Analysis of Partial Differential Equations

Concepción ● Chile ● January 22, 2019

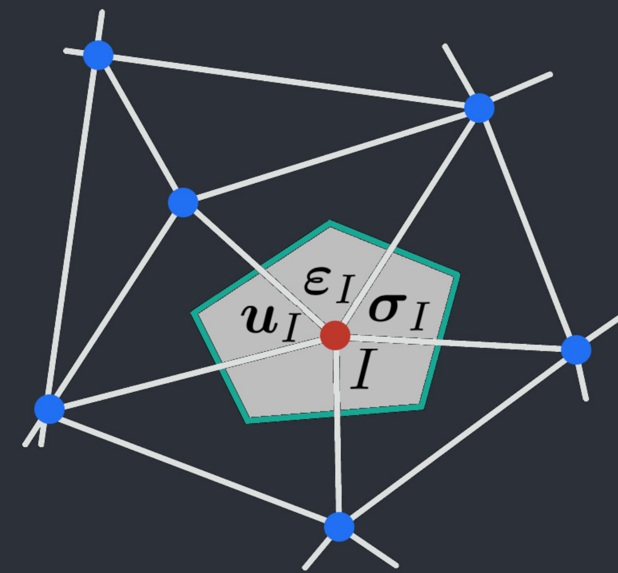
Outline

- Motivation
- Meshfree Basis Functions
- Meshfree Integration Cells
- Nodal Integration Using the Virtual Element Decomposition (NIVED)
- Numerical Tests
- Summary and Outlook

Motivation



Numerical integration of the stiffness matrix



Gauss integration

- Displacements at nodes
- Stresses/Strains at Gauss points

Nodal integration

- Displacements at nodes
- Stresses/Strains at nodes

Puso et al., IJNME, 74(3), 2008:

- Direct integration at nodes leads to instabilities
- A penalty stabilization is added to the stiffness matrix

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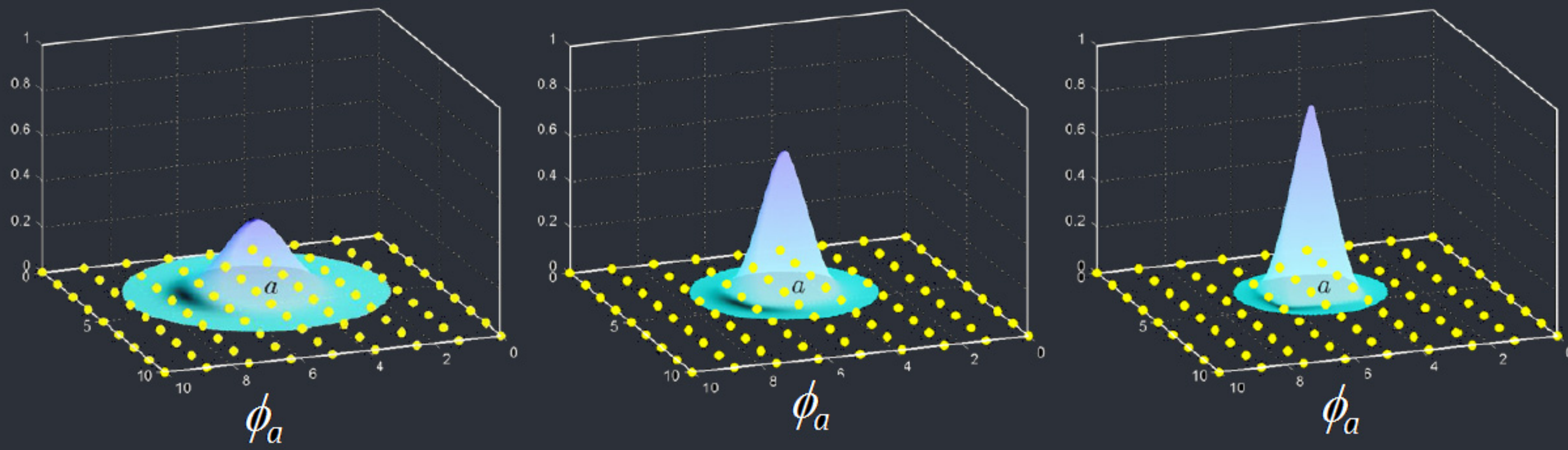
Meshfree Basis Functions

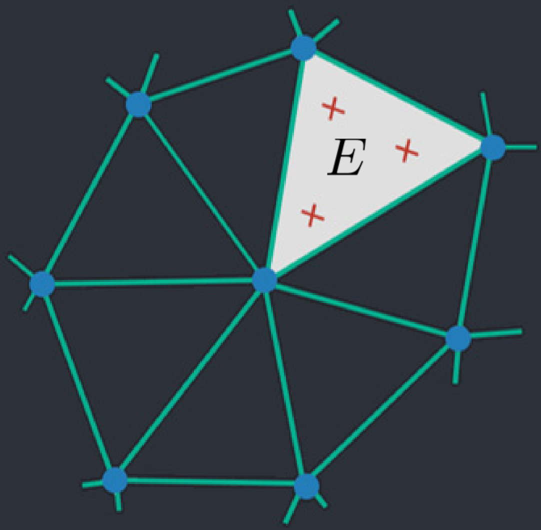


- Rational (nonpolynomial) basis functions
- Linear approximation plus some nonpolynomial terms
- Integration errors affecting consistency and stability

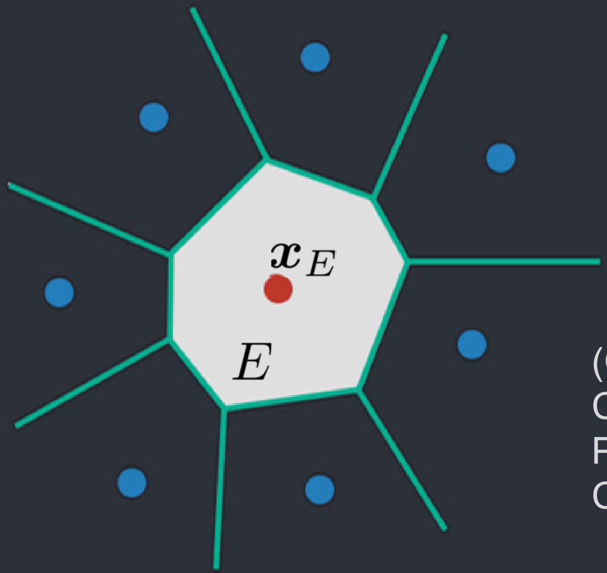
[Belytschko et al., IJNME, 37(2), 1994;
Liu et al., IJNMF, 20(8-9), 1995;
Atluri & Zhu, CM, 22(2), 1998;
De & Bathe, CM, 25(4), 2000;
Sukumar et al., IJNME, 43(5), 1998;
Sukumar, IJNME, 61(12), 2004;
Arroyo & Ortiz, IJNME, 65(13), 2006
… and many others!]

Meshfree Integration Cells

Gauss Integration

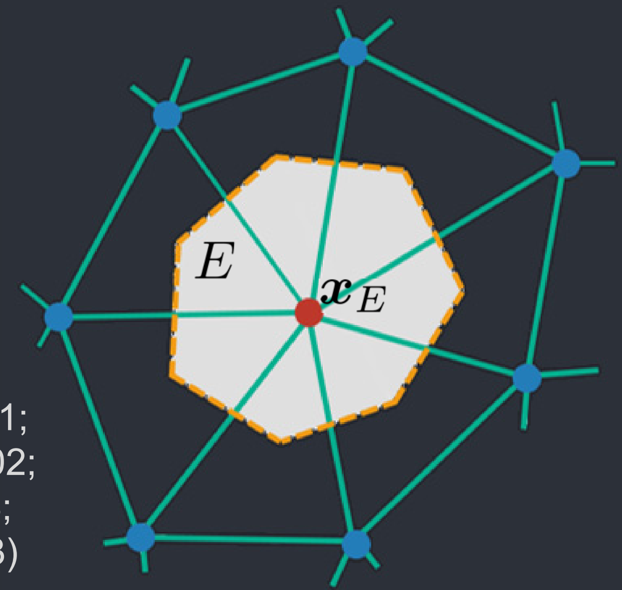


(Dolbow & Belytschko, CM, 23(3), 1999;
Babŭska et al., IJNME, 76(9), 2008;
Ortiz et al., CMAME, 199(25-28), 2010;
Duan et al., IJNME, 92(4), 2012;
Chen et al., IJNME, 95(5), 2013;
Ortiz-Bernardin et al., IJNME, 112(7), 2017)



Nodal Integration

(Chen et al., IJNME, 50(2), 2001;
Chen et al., IJNME, 53(12), 2002;
Puso et al., IJNME, 74(3), 2008;
Chen et al., IJNME, 95(5), 2013)



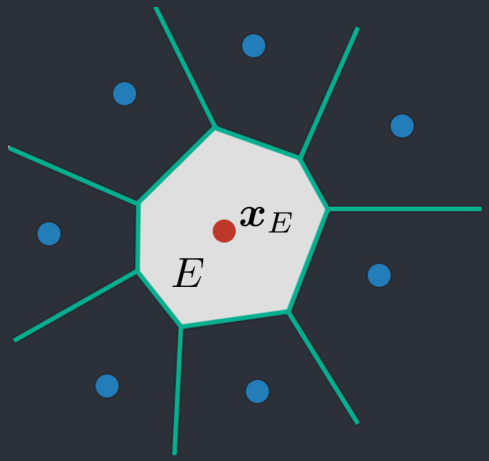
NIVED: Basis and Spaces

Adapted from (O-B et al., IJNME, 112(7), 2017)

Bilinear form (linear elasticity) at the cell level:

$$a_E(\boldsymbol{u}^h, \boldsymbol{v}^h) = \int_E \boldsymbol{\sigma}(\boldsymbol{u}^h) : \nabla \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}^h) = \boldsymbol{D} : \nabla_{\mathrm{S}} \boldsymbol{v}^h$$



Approximation basis:

$$\mu := \{1,\ x,\ y\} \cup \{\text{nonpolynomial terms}\}$$

Displacement spaces:

$\mathcal{P}(E) =$ linear displacements

$\mathcal{H}(E) =$ nonpolynomial displacements

$$\therefore\ \boldsymbol{u}^h, \boldsymbol{v}^h \in \mathcal{W}(E) = \mathcal{P}(E) \oplus \mathcal{H}(E)$$

NIVED: Virtual Element Decomp.

Let $\Pi : \mathcal{W}(E) \rightarrow \mathcal{P}(E)$, $\Pi \boldsymbol{p} = \boldsymbol{p}$, $\forall \boldsymbol{p} \in \mathcal{P}(E)$. Then,

$$\boldsymbol{v}^h = \underbrace{\Pi \boldsymbol{v}^h}_{\in \mathcal{P}} + \underbrace{(\boldsymbol{v}^h - \Pi \boldsymbol{v}^h)}_{\in \mathcal{H}}$$

The projection satisfies the orthogonality condition

$$a_E(\boldsymbol{p}, \boldsymbol{v}^h - \Pi \boldsymbol{v}^h) = 0 \quad \forall \boldsymbol{p} \in \mathcal{P}(E), \quad \boldsymbol{v}^h \in \mathcal{W}(E)$$

which allows writing

$$a_E(\boldsymbol{u}^h, \boldsymbol{v}^h) = a_E(\Pi \boldsymbol{u}^h, \Pi \boldsymbol{v}^h) + a_E(\boldsymbol{u}^h - \Pi \boldsymbol{u}^h, \boldsymbol{v}^h - \Pi \boldsymbol{v}^h)$$

First term gives consistency and second one stability. The stability term can be approximated, which gives

$$a_E^h(\boldsymbol{u}^h, \boldsymbol{v}^h) := a_E(\Pi \boldsymbol{u}^h, \Pi \boldsymbol{v}^h) + s_E(\boldsymbol{u}^h - \Pi \boldsymbol{u}^h, \boldsymbol{v}^h - \Pi \boldsymbol{v}^h)$$

NIVED: Projection Operator

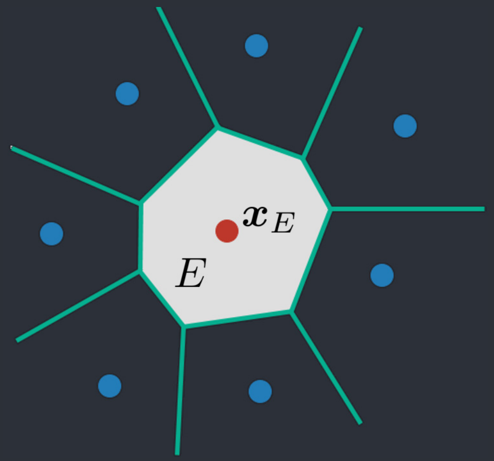
Using the "intrinsic" formula given in (BdV et al., M3A, 24(8), 2014), the projection operator sampled at $\boldsymbol{x}_E$ is

$$\Pi \boldsymbol{v}^h(\boldsymbol{x}_E) = \left(\frac{1}{|E|} \int_E \nabla \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} \right) \cdot (\boldsymbol{x} - \boldsymbol{x}_E) + \boldsymbol{v}^h_E$$

where

$$\frac{1}{|E|} \int_E \nabla \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} = \frac{1}{|E|} \int_E \nabla_{\mathrm{S}} \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} + \frac{1}{|E|} \int_E \nabla_{\mathrm{AS}} \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x}$$

$$\boldsymbol{v}^h_E := \boldsymbol{v}^h(\boldsymbol{x}_E)$$



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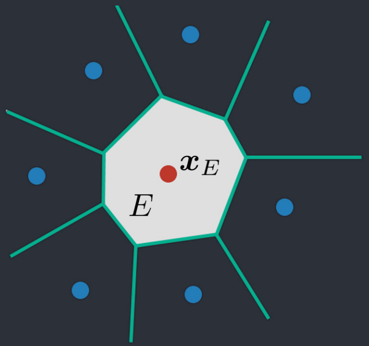
NIVED: Cell Stiffness Matrix

Virtual element decomposition:

$$a_E^h(\boldsymbol{u}^h, \boldsymbol{v}^h) := \underbrace{a_E(\Pi \boldsymbol{u}^h, \Pi \boldsymbol{v}^h)}_{\text{consistency}} + \underbrace{s_E(\boldsymbol{u}^h - \Pi \boldsymbol{u}^h, \boldsymbol{v}^h - \Pi \boldsymbol{v}^h)}_{\text{stability}}$$

Nodal integration: sampling the consistency part at $\boldsymbol{x}_E$ and using the projection operator leads to



$$\begin{aligned} a_E(\Pi \boldsymbol{u}^h, \Pi \boldsymbol{v}^h) &= \int_E \boldsymbol{\sigma}(\Pi \boldsymbol{u}^h) : \nabla \Pi \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} \\ &= \boldsymbol{\sigma}(\Pi \boldsymbol{u}^h(\boldsymbol{x}_E)) : \nabla \Pi \boldsymbol{v}^h(\boldsymbol{x}_E) |E| \\ &= \left(\frac{1}{|E|} \int_E \nabla_{\mathrm{S}} \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} \right) : \boldsymbol{D} : \left(\frac{1}{|E|} \int_E \nabla_{\mathrm{S}} \boldsymbol{u}^h \, \mathrm{d}\boldsymbol{x} \right) |E| \\ &= \boldsymbol{q}^\mathsf{T} \boldsymbol{K}_E^{\mathrm{c}} \boldsymbol{d} \qquad \text{(consistency stiffness)} \end{aligned}$$

NIVED: Cell Stiffness Matrix

Virtual element decomposition:

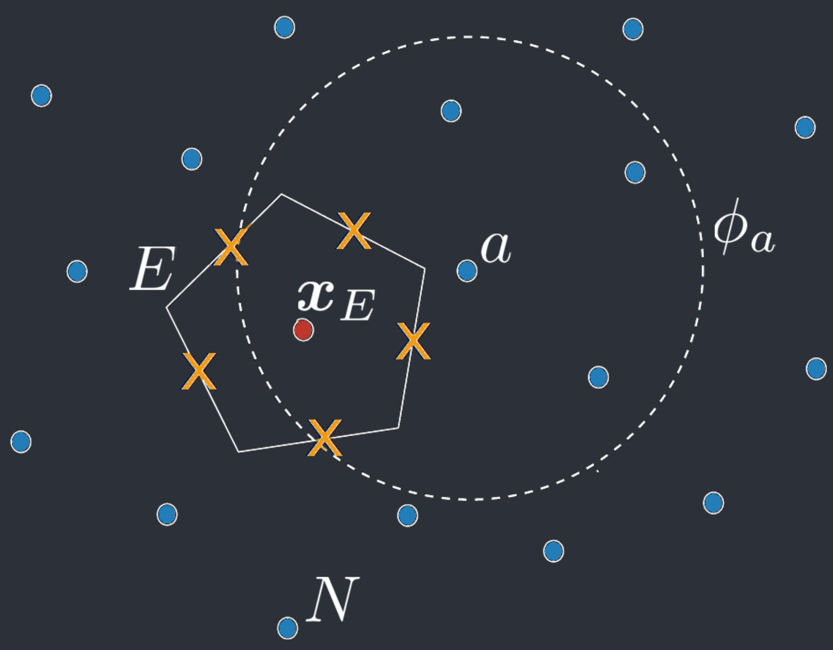
$$a_E^h(\boldsymbol{u}^h, \boldsymbol{v}^h) := \underbrace{a_E(\Pi \boldsymbol{u}^h, \Pi \boldsymbol{v}^h)}_{\text{consistency}} + \underbrace{s_E(\boldsymbol{u}^h - \Pi \boldsymbol{u}^h, \boldsymbol{v}^h - \Pi \boldsymbol{v}^h)}_{\text{stability}}$$

For the stability part choose s_E such that it is positive definite and scales uniformly with the exact bilinear form (e.g.: Gain et al., CMAME, 282, 2014):

$$\begin{aligned} s_E(\Pi \boldsymbol{u}^h, \Pi \boldsymbol{v}^h) &= \boldsymbol{q}^\mathsf{T} (\boldsymbol{I} - \Pi)\, \alpha_E\, \boldsymbol{I}\, (\boldsymbol{I} - \Pi) \boldsymbol{d} \\ &= \boldsymbol{q}^\mathsf{T} \boldsymbol{K}_E^{\mathrm{s}} \boldsymbol{d} \quad \text{(stability stiffness)} \end{aligned}$$

where α_E is an scaling parameter.

NIVED: Implementation



- ● cloud of nodes (including ●) from 1 to N
- ● is the integration point with coordinates $\boldsymbol{x}_E$
- E is the integration cell and the representative nodal volume
- X are auxiliary integration points on the boundary of the cell
- Contribution: support of ϕ_a touches the cell

NIVED: Implementation (Cont'd)

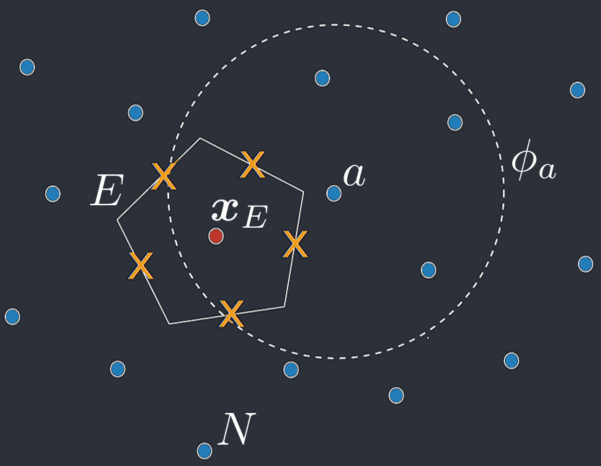
Everything reduces to the computation of the constant strain over the element:

$$\frac{1}{|E|}\int_E \boldsymbol{\nabla}\boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} = \frac{1}{|E|}\int_E \boldsymbol{\nabla}_{\mathrm{S}}\boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} + \frac{1}{|E|}\int_E \boldsymbol{\nabla}_{\mathrm{AS}}\boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x}$$

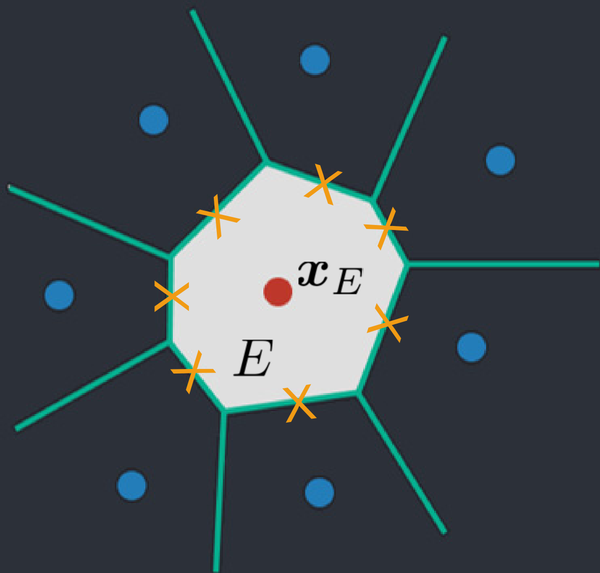
By applying the divergence theorem, these integrals are evaluated on the boundary of the cell:

$$\frac{1}{|E|}\int_E \boldsymbol{\nabla}_{\mathrm{S}}\boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} = \frac{1}{2|E|}\int_{\partial E} \left(\boldsymbol{v}^h \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{v}^h\right) \, \mathrm{d}s$$

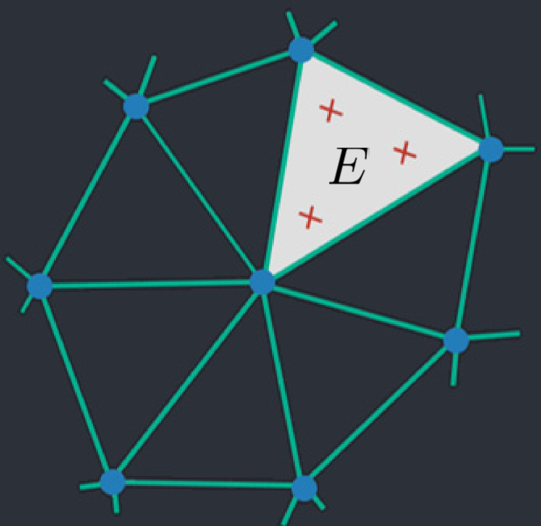
$$\frac{1}{|E|}\int_E \boldsymbol{\nabla}_{\mathrm{AS}}\boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} = \frac{1}{2|E|}\int_{\partial E} \left(\boldsymbol{v}^h \otimes \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{v}^h\right) \, \mathrm{d}s$$



Numerical Tests: Integration Cells



Meshes use the same node set



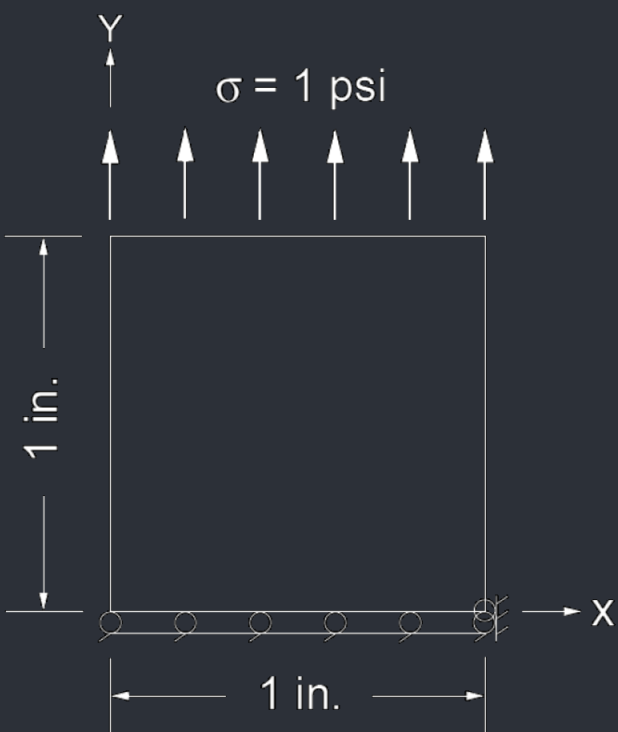
NIVED

1-pt per edge
(basis function only)

MEM

Internal Gauss points:
1-pt, 3-pt, 6-pt, 12-pt
(basis functions and derivatives)

Test: Patch Test



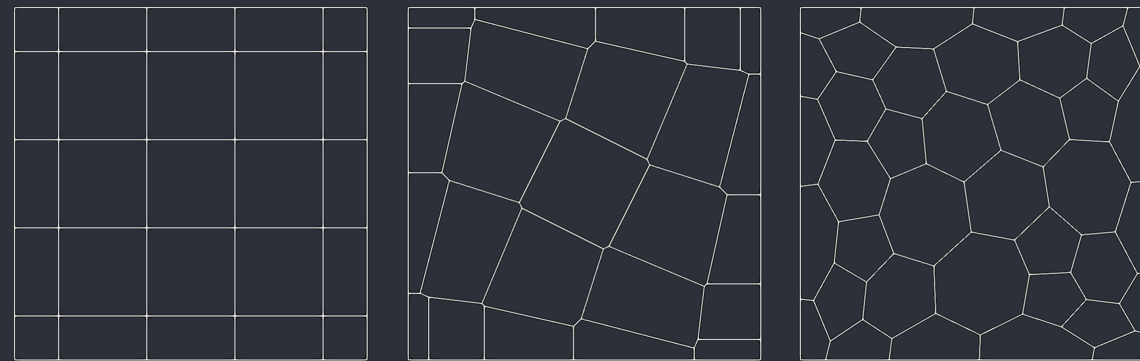


Table 1: Relative error in the H^1 seminorm

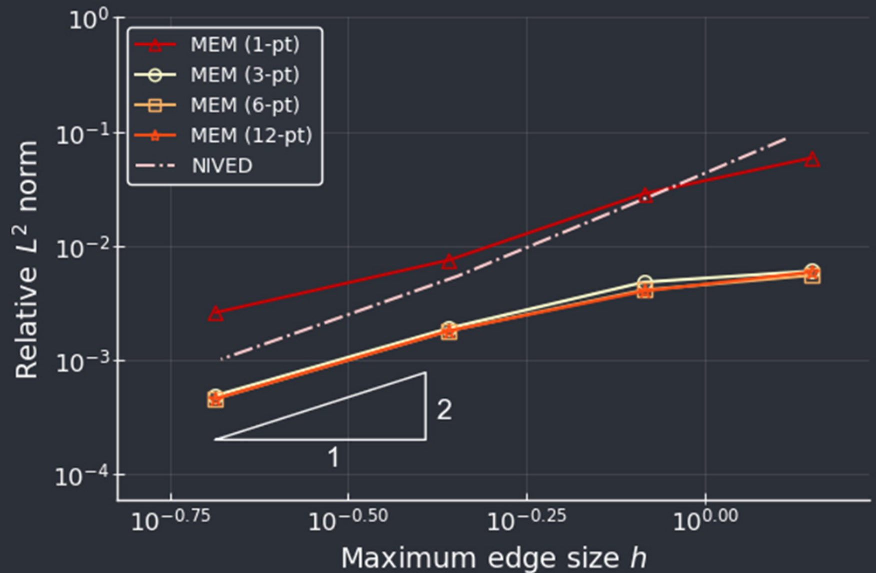
Method	Gauss rule	Regular	Distorted	Unstructured
MEM	1-pt	1.4×10^{-2}	5.4×10^{-2}	2.5×10^{-2}
MEM	3-pt	2.6×10^{-3}	5.3×10^{-3}	4.8×10^{-3}
MEM	6-pt	5.4×10^{-5}	1.9×10^{-3}	1.3×10^{-3}
MEM	12-pt	2.3×10^{-7}	7.7×10^{-4}	4.5×10^{-4}
NIVED	1-pt/edge	3.6×10^{-15}	5.2×10^{-15}	7.8×10^{-15}

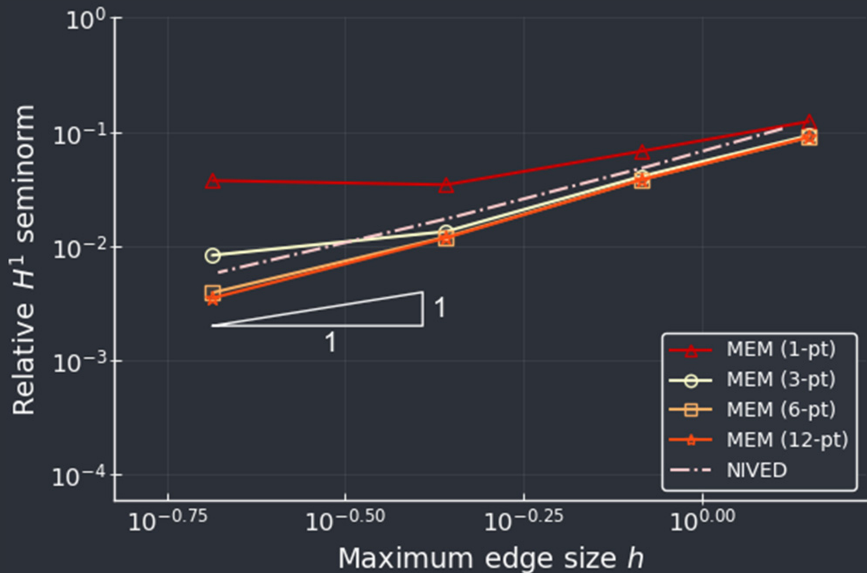
Table 2: Relative error in the L^2 norm

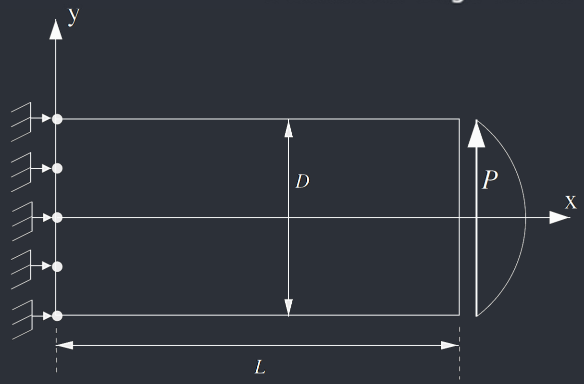
Method	Gauss rule	Regular	Distorted	Unstructured
MEM	1-pt	1.0×10^{-2}	2.0×10^{-2}	1.7×10^{-2}
MEM	3-pt	2.3×10^{-3}	1.6×10^{-3}	1.6×10^{-3}
MEM	6-pt	5.0×10^{-5}	8.0×10^{-4}	1.2×10^{-3}
MEM	12-pt	2.2×10^{-7}	3.0×10^{-4}	5.0×10^{-4}
NIVED	1-pt/edge	3.6×10^{-15}	4.1×10^{-15}	2.5×10^{-15}

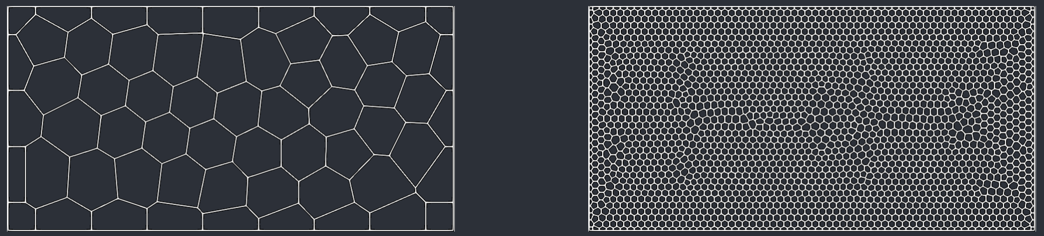
Test: Cantilever Beam

Convergence







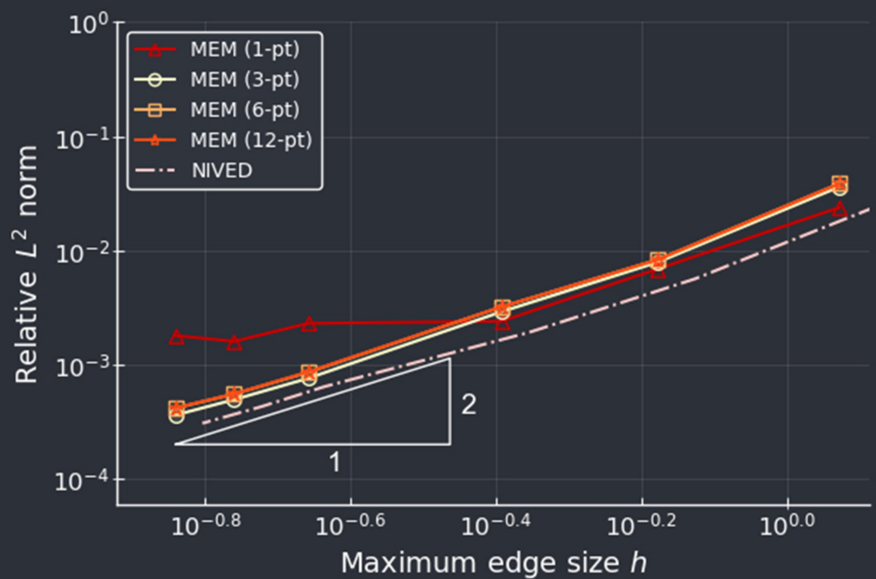


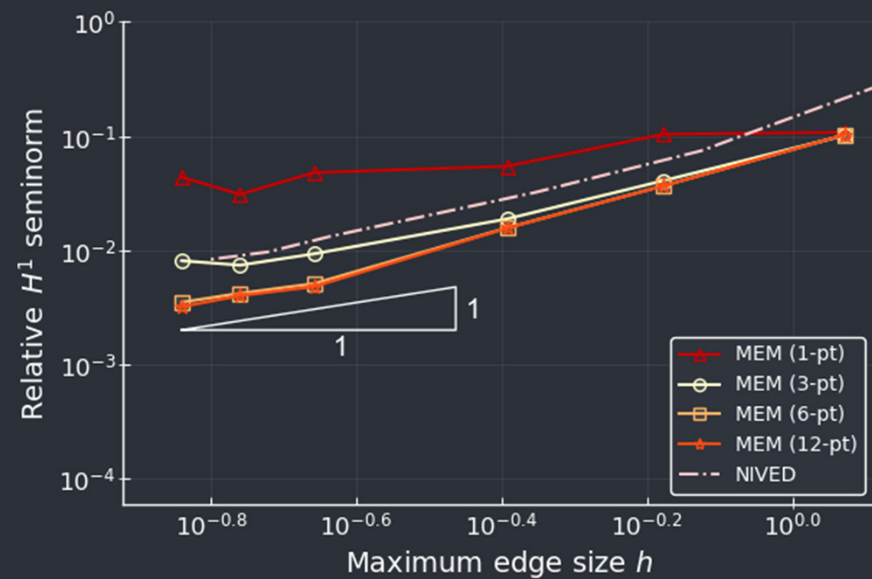
coarsest

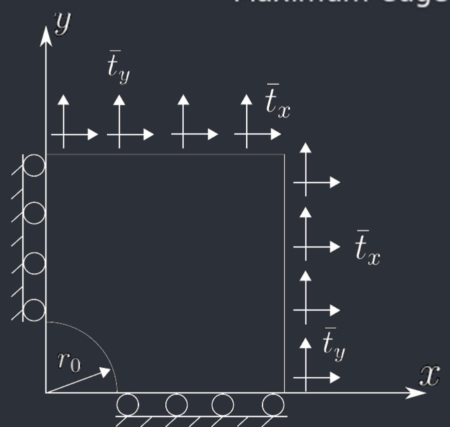
finest

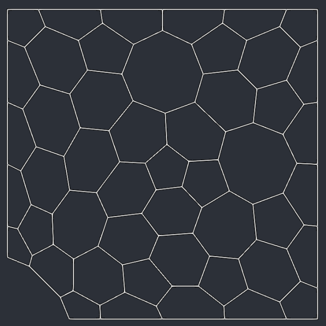
Test: Plate With a Hole

Convergence









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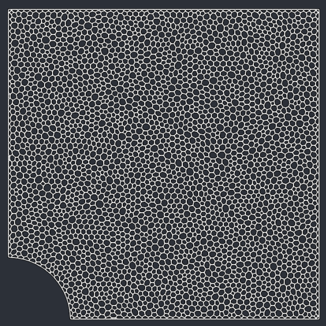

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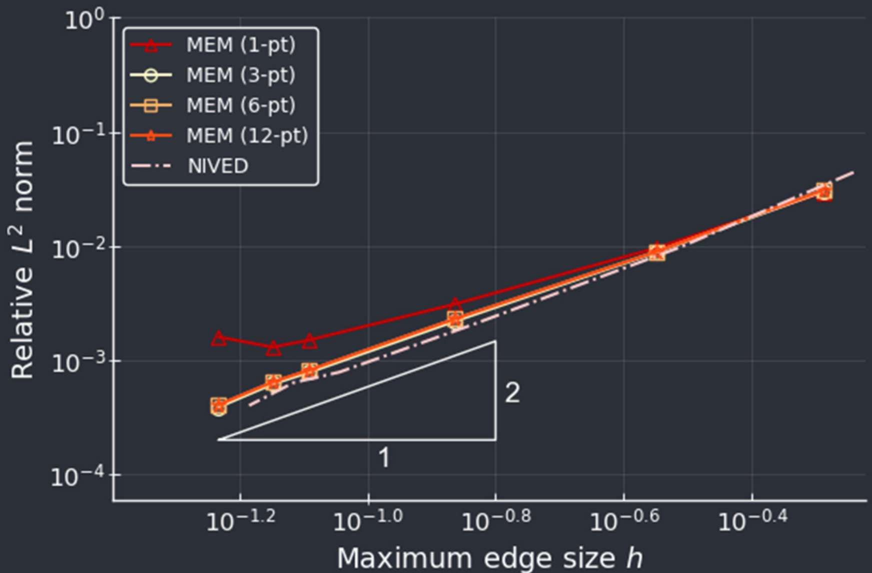
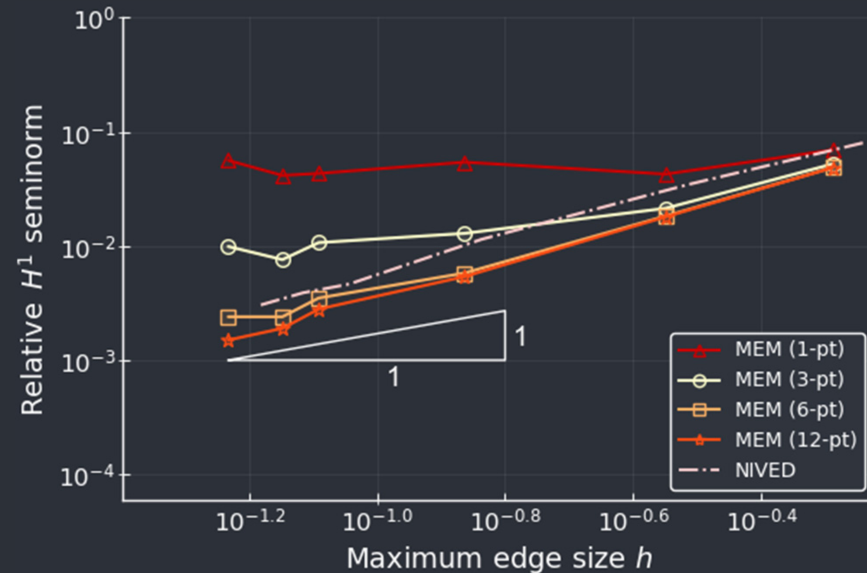
Test: Manufactured Problem

Convergence

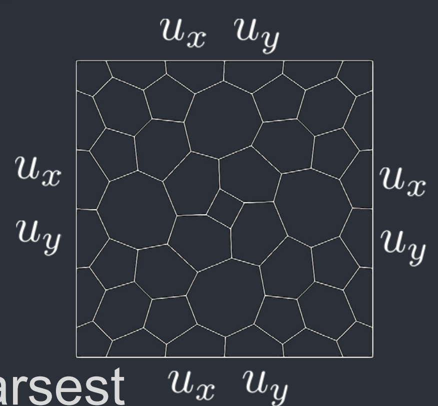




Body force:

$$b_x = \sin x \cos y D_{11} - e^x e^y D_{12} - (e^x e^y - \sin x \cos y) D_{33}$$

$$b_y = \cos x \sin y D_{21} - e^x e^y D_{22} - (e^x e^y - \cos x \sin y) D_{33}$$



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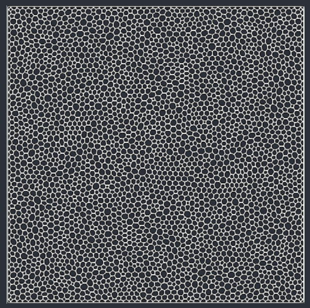


finest

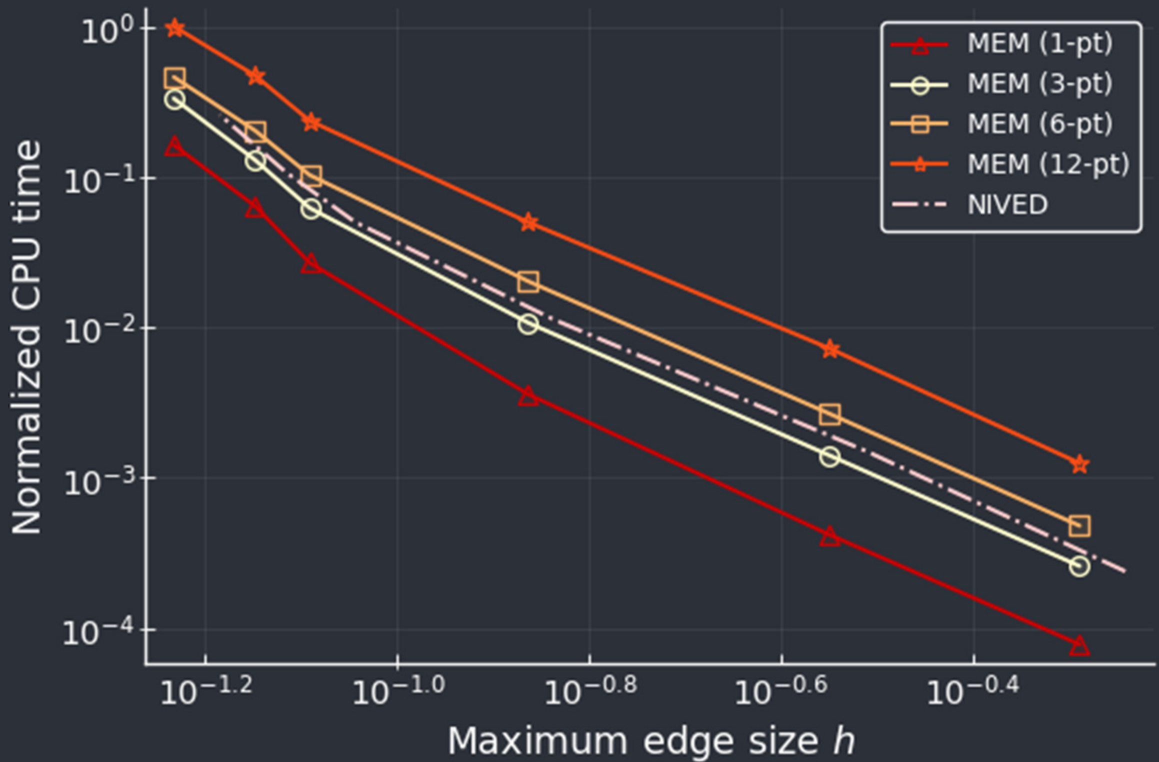
Dirichlet BCs:

$$u_x = \sin(x)\cos(y)$$

$$u_y = e^x e^y$$

Test: Manufactured Problem (Cont'd)

Timing



Normalized CPU time = $T/T_{\max}$

Summary and Outlook

- Developed NIVED (Nodal Integration – VED) for linear elasticity
- VED provides consistency (patch test satisfaction) and optimal convergence to the nodal integration method
- Computational cost: NIVED similar to 3-pt Gauss rule
- NIVED for nonlinear problems (elastoplasticity) is under development
- Outlook: large deformations, 3D simulations

Journal Acronyms

- M3A: Mathematical Models and Methods in Applied Sciences
- CMAME: Computer Methods in Applied Mechanics and Engineering
- IJNME: International Journal for Numerical Methods in Engineering
- IJNMF: International Journal for Numerical Methods in Fluids
- CM: Computational Mechanics