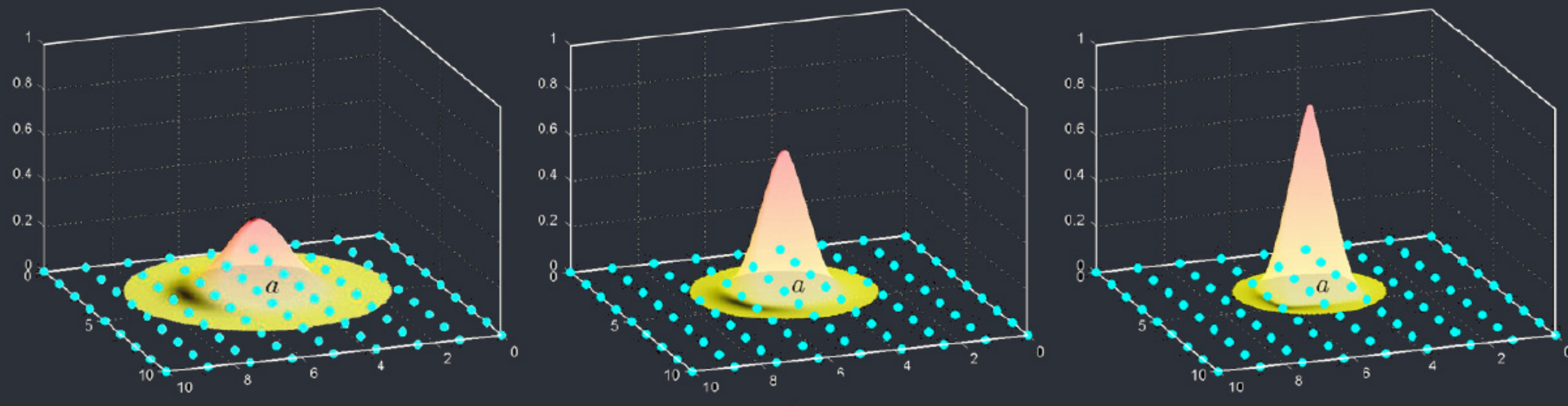
A MESHFREE NODAL INTEGRATION METHOD FOR ELASTIC AND ELASTOPLASTIC APPLICATIONS USING THE VIRTUAL ELEMENT DECOMPOSITION



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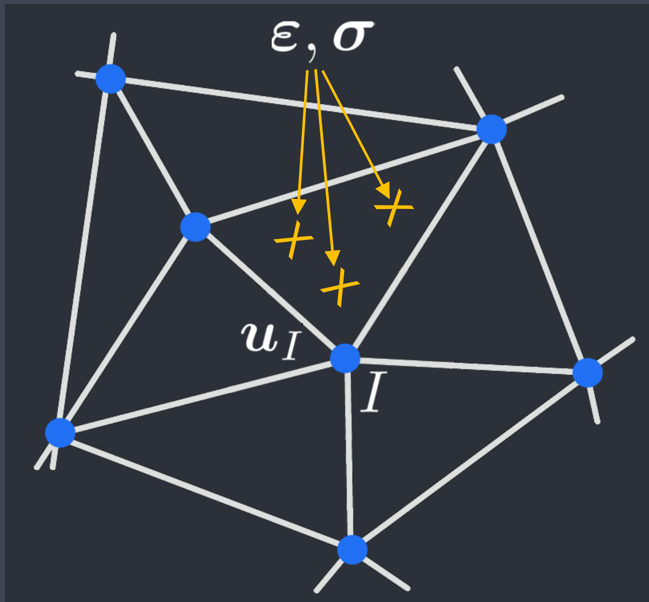
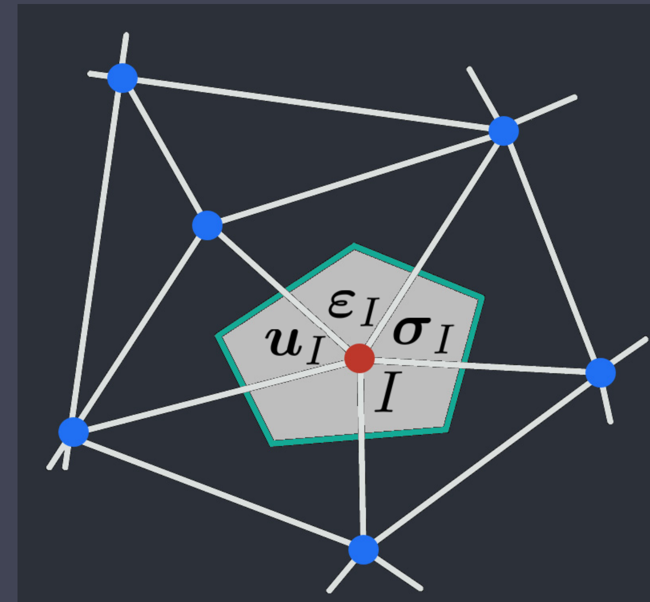
Outline

- Motivation
- Meshfree Basis Functions
- Nodal Integration Using the Virtual Element Decomposition (NIVED)
- Numerical Tests
- Summary and Outlook

Motivation



Numerical integration of the stiffness matrix



Gauss integration

- Displacements at nodes
- Stresses/Strains at Gauss points

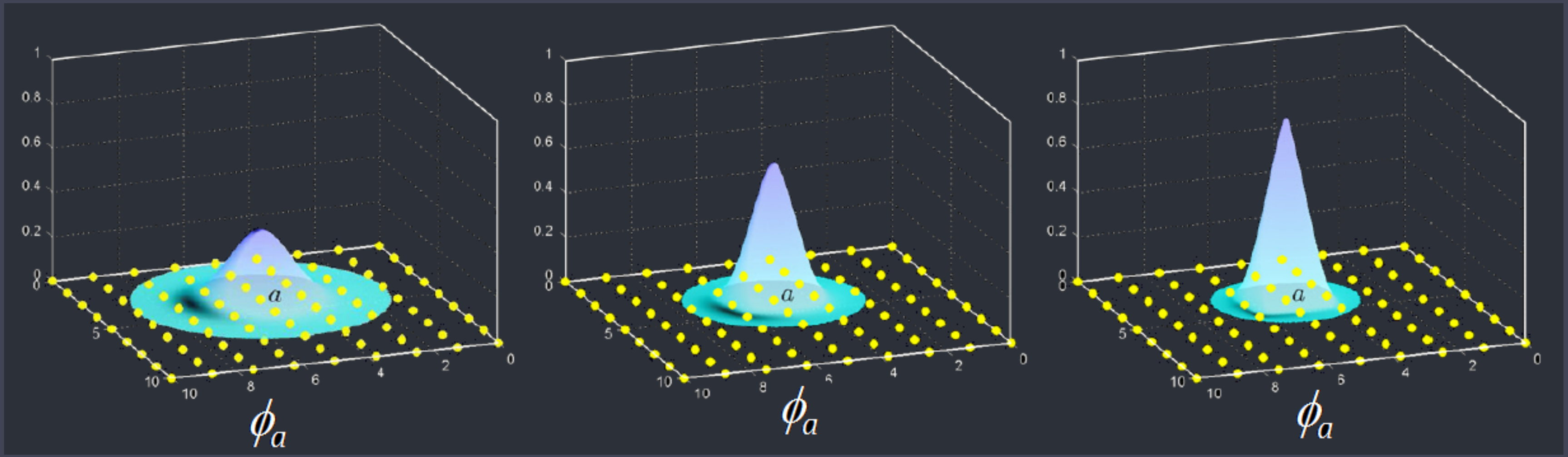
Nodal integration

- Displacements at nodes
- Stresses/Strains at nodes

Chen et al., IJNME, 50(2), 2001 - IJNME, 53(12), 2002; Puso et al., IJNME, 74(3), 2008:

- Direct integration at nodes leads to instabilities
- A penalty stabilization is added to the stiffness matrix

Meshfree Basis Functions



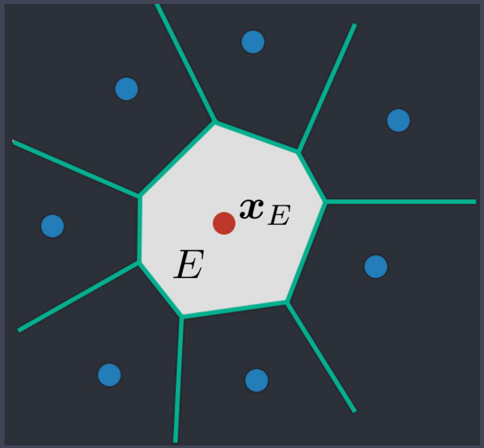
- Less sensitive to mesh distortion than FEM basis functions
- Nonpolynomial basis functions
- Linear approximation plus some nonpolynomial terms
- Integration errors affecting consistency and stability

Belytschko et al., IJNME, 37(2), 1994;
Liu et al., IJNMF, 20(8-9), 1995;
Atluri & Zhu, CM, 22(2), 1998;
De & Bathe, CM, 25(4), 2000;
Sukumar et al., IJNME, 43(5), 1998;
Sukumar, IJNME, 61(12), 2004;
Arroyo & Ortiz, IJNME, 65(13), 2006
… and many others!

Linear NIVED: Bilinear Form

We want to evaluate the bilinear form of the linear elasticity boundary value problem,

$$a_E(\boldsymbol{u}^h, \boldsymbol{v}^h) = \int_E \boldsymbol{\sigma}(\boldsymbol{u}^h) : \nabla \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x}$$



at the node $\boldsymbol{x}_E$ of a representative nodal cell using a nodal projection operator $\mathit{\Pi}_E$ and a stabilization term (both to be defined), which gives

$$\boxed{a_E(\boldsymbol{u}^h, \boldsymbol{v}^h) = |E| \boldsymbol{\sigma}(\mathit{\Pi}_E \boldsymbol{u}^h) : \nabla \mathit{\Pi}_E \boldsymbol{v}^h + \text{stabilization}}$$

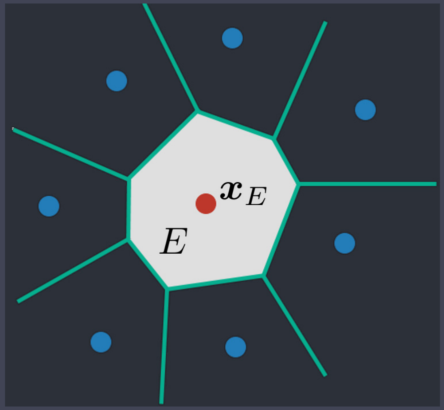
Linear NIVED: Virtual Element Decomposition

We use the virtual element decomposition (BdV et al., M3A, 24(8), 2014):

$$\boxed{a_E^h(\boldsymbol{u}^h, \boldsymbol{v}^h) := a_E(\Pi_E \boldsymbol{u}^h, \Pi_E \boldsymbol{v}^h) + s_E(\boldsymbol{u}^h - \Pi_E \boldsymbol{u}^h, \boldsymbol{v}^h - \Pi_E \boldsymbol{v}^h)}$$

where the first term gives consistency and second one stability, and Π_E is a nodal projection operator that for linear displacements is given by the "intrinsic" formula (BdV et al., M3A, 24(8), 2014):

$$\Pi_E \boldsymbol{v}^h = \left(\frac{1}{|E|} \int_E \boldsymbol{\nabla} \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} \right) \cdot (\boldsymbol{x} - \boldsymbol{x}_E) + \boldsymbol{v}^h(\boldsymbol{x}_E)$$



Linear NIVED: Virtual Element Decomp. (Cont'd)

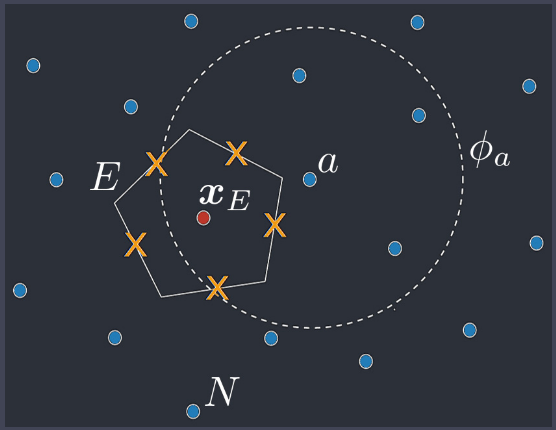
Everything reduces to the computation of the constant strain over the element:

$$\frac{1}{|E|}\int_E \nabla \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} = \frac{1}{|E|}\int_E \nabla_{\mathrm{S}} \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} + \frac{1}{|E|}\int_E \nabla_{\mathrm{AS}} \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x}$$

By applying the divergence theorem, these integrals are evaluated on the boundary of the cell (def.: nodal strains):

$$\boldsymbol{\varepsilon}_E^h = \frac{1}{|E|}\int_E \nabla_{\mathrm{S}} \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} = \frac{1}{2|E|}\int_{\partial E} \left(\boldsymbol{v}^h \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{v}^h\right) \, \mathrm{d}s$$

$$\boldsymbol{\omega}_E^h = \frac{1}{|E|}\int_E \nabla_{\mathrm{AS}} \boldsymbol{v}^h \, \mathrm{d}\boldsymbol{x} = \frac{1}{2|E|}\int_{\partial E} \left(\boldsymbol{v}^h \otimes \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{v}^h\right) \, \mathrm{d}s$$



Linear NIVED: Cell Stiffness Matrix

Virtual element decomposition (VED):

$$a_E^h(\boldsymbol{u}^h, \boldsymbol{v}^h) := \underbrace{a_E(\mathit{\Pi}_E\boldsymbol{u}^h, \mathit{\Pi}_E\boldsymbol{v}^h)}_{\text{consistency}} + \underbrace{s_E(\boldsymbol{u}^h - \mathit{\Pi}_E\boldsymbol{u}^h, \boldsymbol{v}^h - \mathit{\Pi}_E\boldsymbol{v}^h)}_{\text{stability}}$$

Nodal integration: sampling the consistency part at $\boldsymbol{x}_E$ by using the nodal projection operator leads to

$$\begin{aligned} a_E(\mathit{\Pi}_E\boldsymbol{u}^h, \mathit{\Pi}_E\boldsymbol{v}^h) &= \boldsymbol{\sigma}(\mathit{\Pi}_E\boldsymbol{u}^h) : \boldsymbol{\nabla}\mathit{\Pi}_E\boldsymbol{v}^h |E| \\ &= \left(\frac{1}{|E|}\int_E \boldsymbol{\nabla}_{\mathrm{S}}\boldsymbol{v}^h \,\mathrm{d}\boldsymbol{x}\right) : \boldsymbol{D} : \left(\frac{1}{|E|}\int_E \boldsymbol{\nabla}_{\mathrm{S}}\boldsymbol{u}^h \,\mathrm{d}\boldsymbol{x}\right) |E| \\ &= \boldsymbol{q}^{\mathsf{T}} \boldsymbol{K}_E^{\mathrm{c}} \boldsymbol{d} \quad \text{(consistency stiffness)} \end{aligned}$$

Virtual element decomposition (VED):

$$a_E^h(\boldsymbol{u}^h, \boldsymbol{v}^h) := \underbrace{a_E(\mathit{\Pi}_E \boldsymbol{u}^h, \mathit{\Pi}_E \boldsymbol{v}^h)}_{\text{consistency}} + \underbrace{s_E(\boldsymbol{u}^h - \mathit{\Pi}_E \boldsymbol{u}^h, \boldsymbol{v}^h - \mathit{\Pi}_E \boldsymbol{v}^h)}_{\text{stability}}$$

For the stability part the approximate s_E is chosen in the spirit of the "*D*-recipe" (BdV et al., CMA, 74(5), 2017):

$$\begin{aligned} s_E(\boldsymbol{u}^h - \mathit{\Pi}_E \boldsymbol{u}^h, \boldsymbol{v}^h - \mathit{\Pi}_E \boldsymbol{v}^h) &= \boldsymbol{q}^\top (\boldsymbol{I} - \mathit{\Pi}_E)\, \mathrm{diag}\big(\mathrm{diag}(\boldsymbol{K}_E^{\mathrm{c}})\big)\, (\boldsymbol{I} - \mathit{\Pi}_E) \boldsymbol{d} \\ &= \boldsymbol{q}^\top \boldsymbol{K}_E^{\mathrm{s}}\, \boldsymbol{d} \quad \text{(stability stiffness)} \end{aligned}$$

Nonlinear NIVED: Incremental Solution

VED representation of the Virtual Work: given $\boldsymbol{\alpha}_n = \boldsymbol{\alpha}(t_n)$ find $\boldsymbol{u}^h_{n+1}$ such that

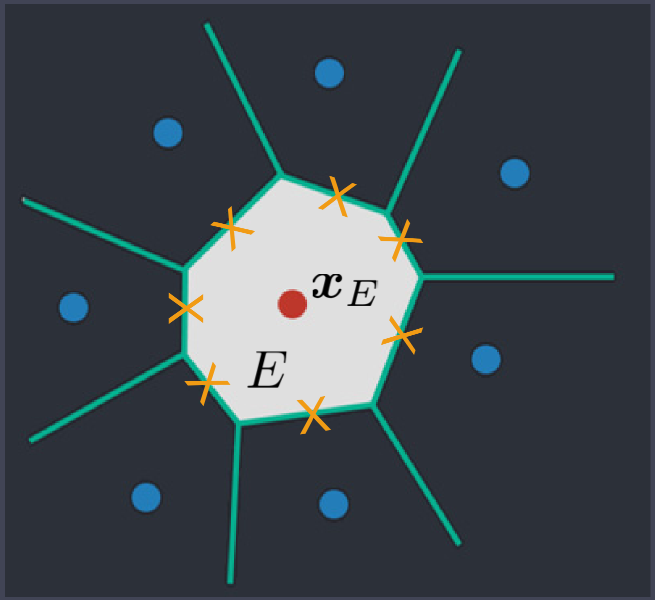
$$\boxed{\begin{aligned}&|E|\,\hat{\boldsymbol{\sigma}}_E\big(\boldsymbol{\alpha}_n, (\boldsymbol{\varepsilon}^h_E)_{n+1}\big) : \delta\boldsymbol{\varepsilon}^h_E + \int_E \hat{\boldsymbol{\sigma}}_E\big(\boldsymbol{\alpha}_n, (\boldsymbol{\varepsilon}^h - \boldsymbol{\varepsilon}^h_E)_{n+1}\big) : \delta(\boldsymbol{\varepsilon}^h - \boldsymbol{\varepsilon}^h_E)\, d\boldsymbol{x} \\ &- |E|\,\boldsymbol{b}^h_{n+1}(\boldsymbol{x}_E)\cdot\delta\boldsymbol{u}^h - |S|\,\bar{\boldsymbol{t}}_{n+1}(\boldsymbol{x}_S)\cdot\delta\boldsymbol{u}^h = 0\end{aligned}}$$

where $\boldsymbol{x}_S$ is a node on the Neumann boundary.

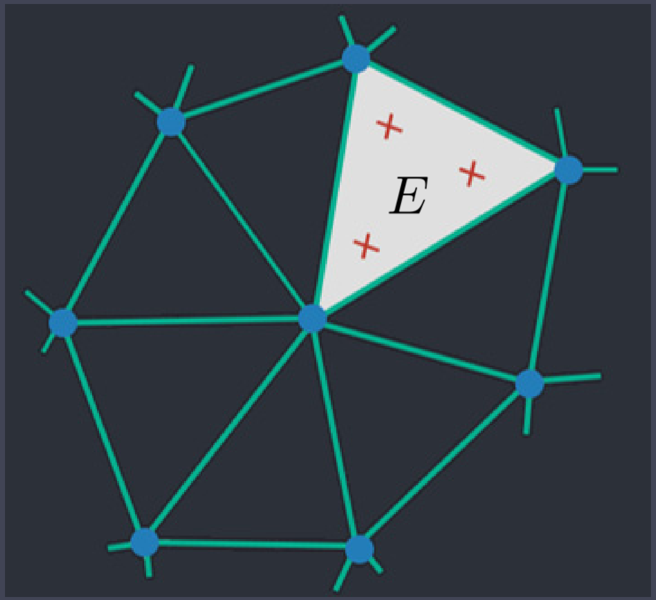
Linearization using the small strain assumption:

$$\begin{aligned}&(\delta\boldsymbol{\varepsilon}^h_E)^\mathsf{T}\boldsymbol{D}_t\Delta\boldsymbol{\varepsilon}^h_E\,|E| + (\boldsymbol{I}-\mathit{\Pi}_E)^\mathsf{T} s_E(\Delta\boldsymbol{u}^h_{n+1}, \delta\boldsymbol{u}^h)(\boldsymbol{I}-\mathit{\Pi}_E) \\ &= -\Big\{(\delta\boldsymbol{\varepsilon}^h_E)^\mathsf{T}(\boldsymbol{\sigma}_E)_{n+1}|E| + (\boldsymbol{I}-\mathit{\Pi}_E)^\mathsf{T} s_E(\boldsymbol{u}^h_{n+1}, \delta\boldsymbol{u}^h)(\boldsymbol{I}-\mathit{\Pi}_E) \\ &\qquad - (\delta\boldsymbol{u}^h)^\mathsf{T}\boldsymbol{b}^h_{n+1}(\boldsymbol{x}_E)|E| \\ &\qquad - (\delta\boldsymbol{u}^h)^\mathsf{T}\bar{\boldsymbol{t}}_{n+1}(\boldsymbol{x}_S)|E|\Big\}\end{aligned}$$

Numerical Tests (Linear NIVED): Integration Cells



Meshes use the same node set



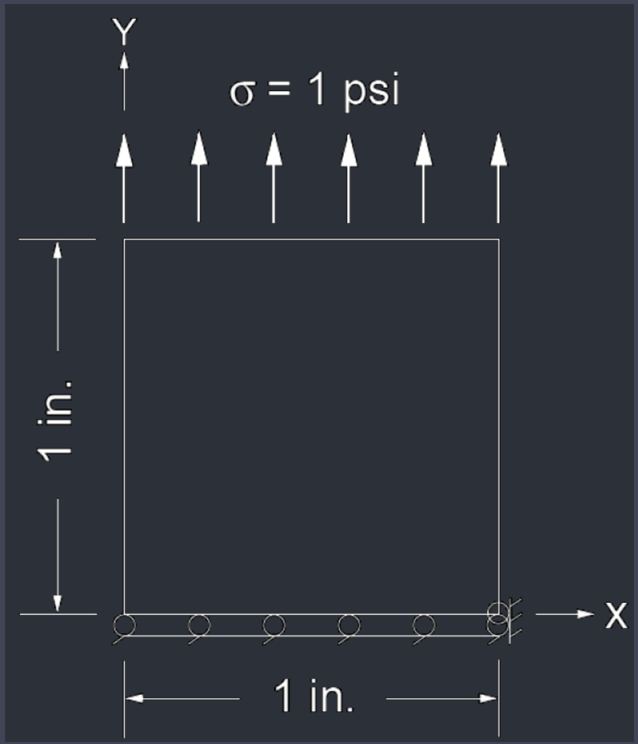
NIVED

1-pt per edge
(basis function only)

MEM

Internal points:
1-pt, 3-pt, 6-pt, 12-pt
(basis functions and derivatives)

Linear NIVED: Patch Test



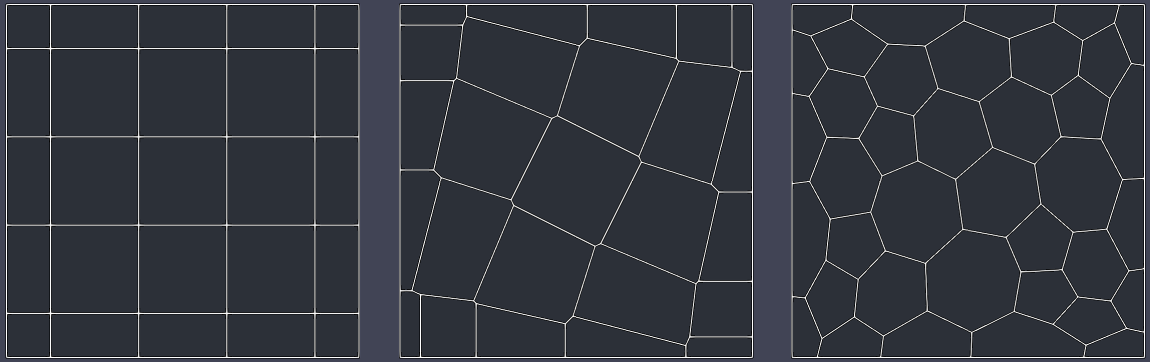


Table 1: Relative error in the H^1 seminorm

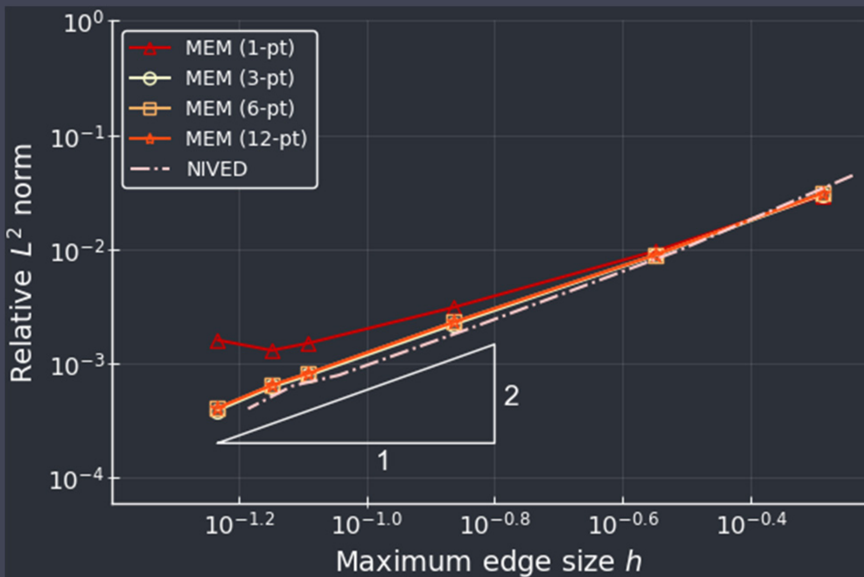
Method	Gauss rule	Regular	Distorted	Unstructured
MEM	1-pt	1.4×10^{-2}	5.4×10^{-2}	2.5×10^{-2}
MEM	3-pt	2.6×10^{-3}	5.3×10^{-3}	4.8×10^{-3}
MEM	6-pt	5.4×10^{-5}	1.9×10^{-3}	1.3×10^{-3}
MEM	12-pt	2.3×10^{-7}	7.7×10^{-4}	4.5×10^{-4}
NIVED	1-pt/edge	3.6×10^{-15}	5.2×10^{-15}	7.8×10^{-15}

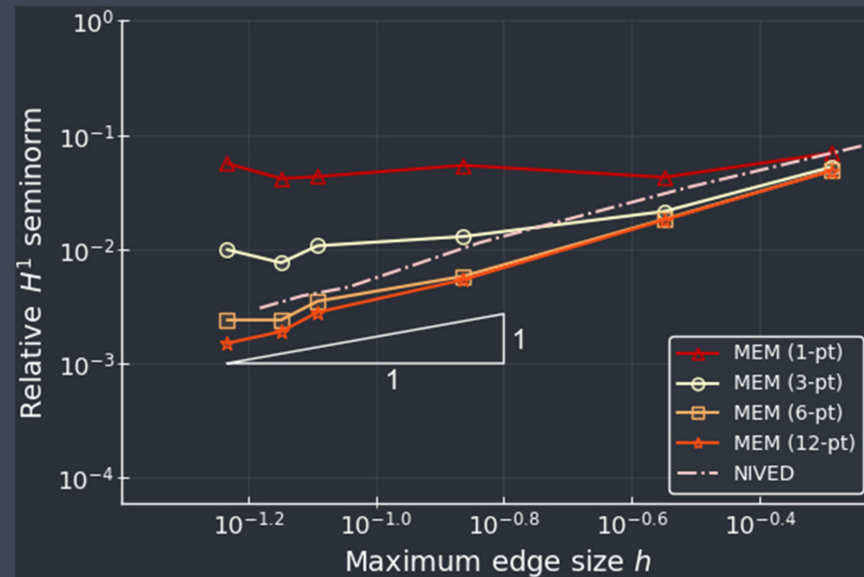
Table 2: Relative error in the L^2 norm

Method	Gauss rule	Regular	Distorted	Unstructured
MEM	1-pt	1.0×10^{-2}	2.0×10^{-2}	1.7×10^{-2}
MEM	3-pt	2.3×10^{-3}	1.6×10^{-3}	1.6×10^{-3}
MEM	6-pt	5.0×10^{-5}	8.0×10^{-4}	1.2×10^{-3}
MEM	12-pt	2.2×10^{-7}	3.0×10^{-4}	5.0×10^{-4}
NIVED	1-pt/edge	3.6×10^{-15}	4.1×10^{-15}	2.5×10^{-15}

Linear NIVED: Manufactured Problem

Convergence

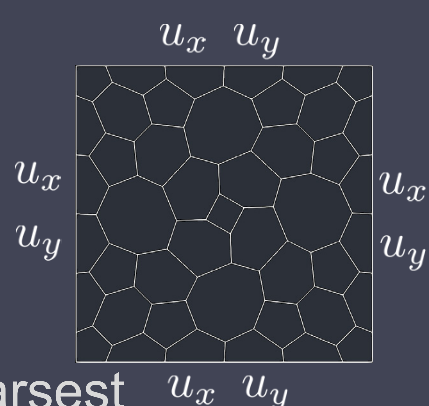




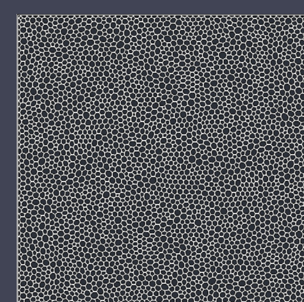
Body force:

$$b_x = \sin x \cos y D_{11} - e^x e^y D_{12} - (e^x e^y - \sin x \cos y) D_{33}$$

$$b_y = \cos x \sin y D_{21} - e^x e^y D_{22} - (e^x e^y - \cos x \sin y) D_{33}$$



coarsest



finest

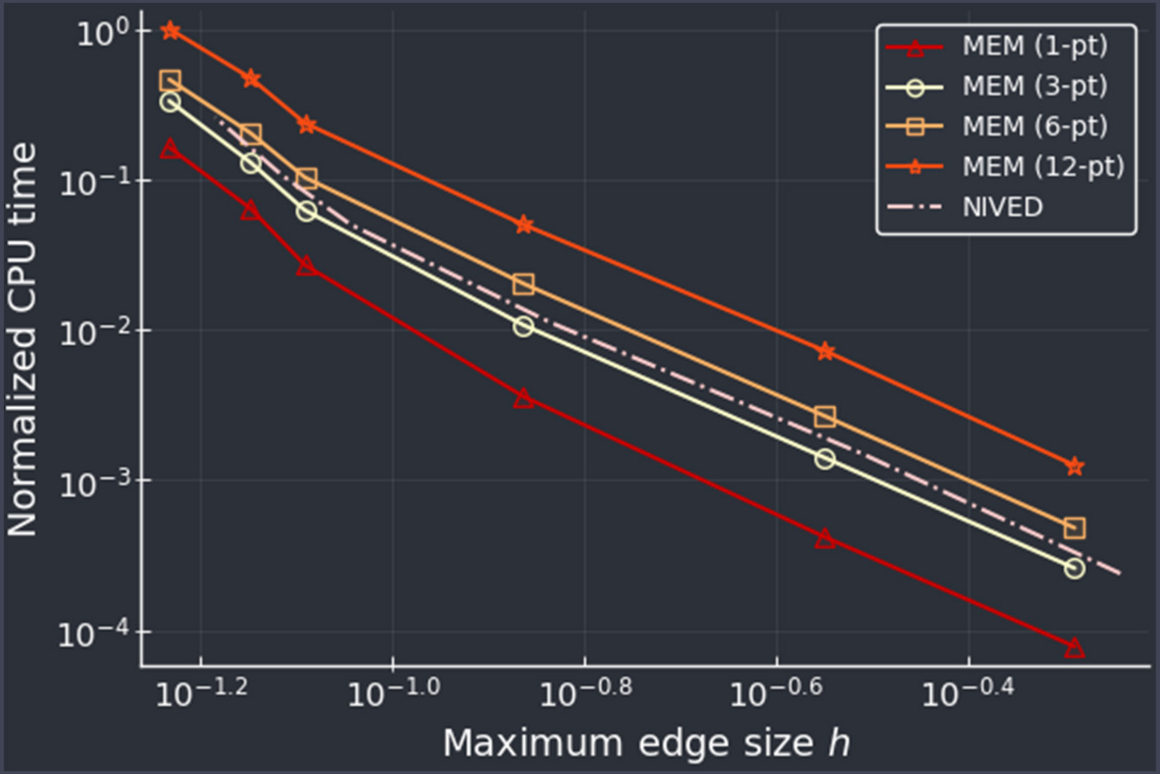
Dirichlet BCs:

$$u_x = \sin(x)\cos(y)$$

$$u_y = e^x e^y$$

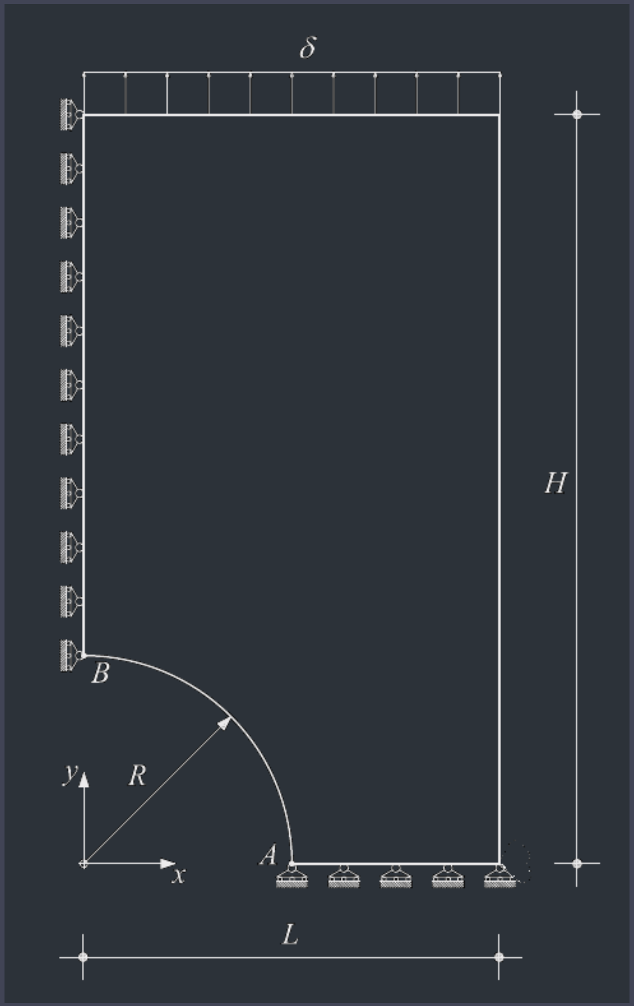
Linear NIVED: Manufactured Problem (Cont'd)

Timing



Normalized CPU time = $T/T_{\max}$

Nonlinear NIVED: Perforated Plate



Dimensions:

- H = 180 mm
- L = 100 mm
- R = 50 mm

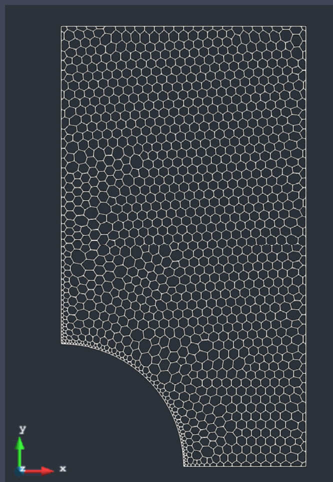
Material:

- J2-Plasticity / Plane strain
- Huber-Mises-Hencky yield function
- Mixed isotropic and kinematic hardening
- E = 70 MPa
- $\nu = 0.2$
- $H_i = 10$ MPa
- $H_k = 10$ MPa
- $\sigma_{y0} = 0.8$ MPa

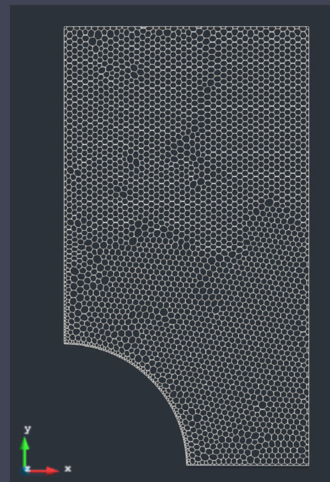
Top displacement:

- $\delta = 10$ mm

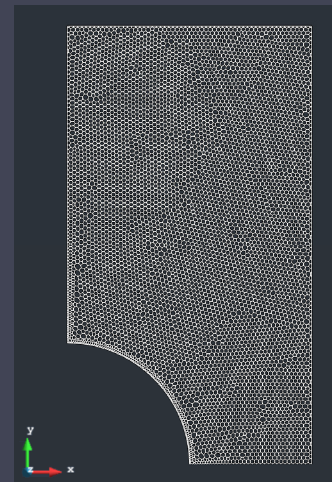
Nonlinear NIVED: Perforated Plate (Cont'd)



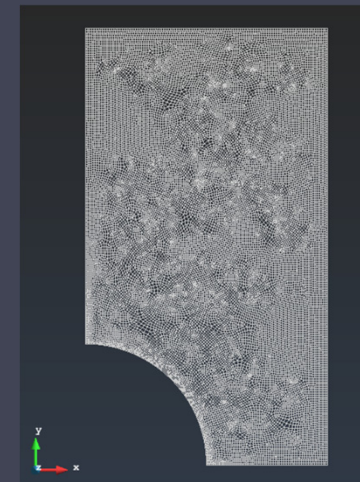
NIVED
(2624 dofs)



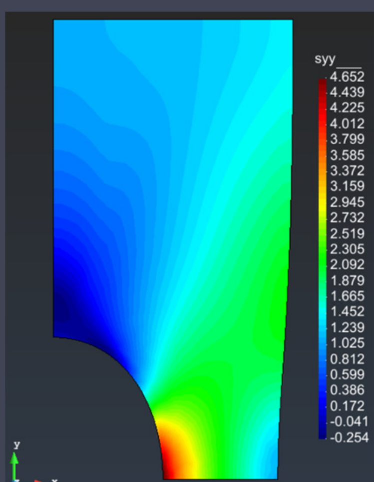
NIVED
(6314 dofs)



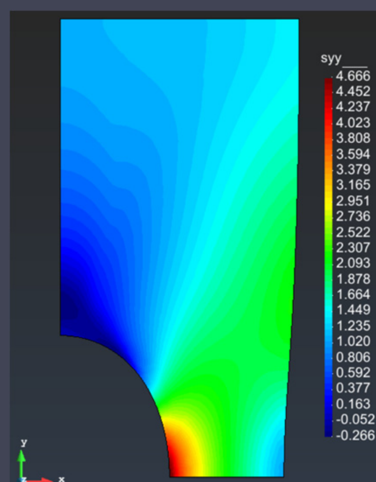
NIVED
(13458 dofs)



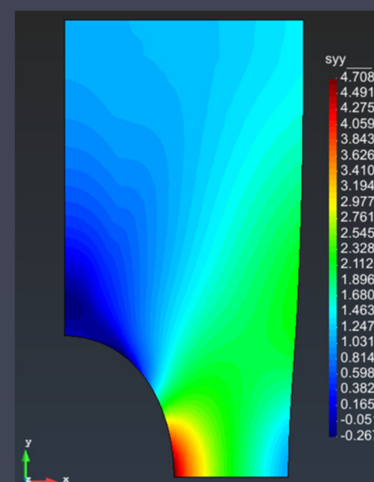
FEM-Q4
(49284 dofs)



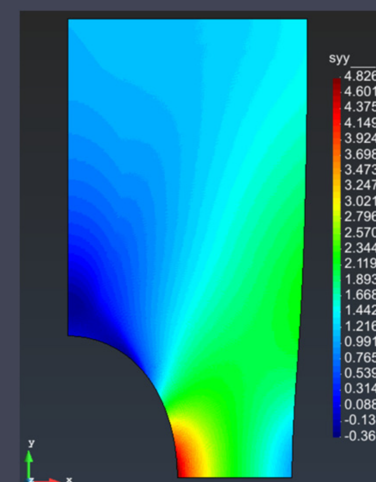
syy max. = 4.652 MPa



syy max. = 4.666 MPa

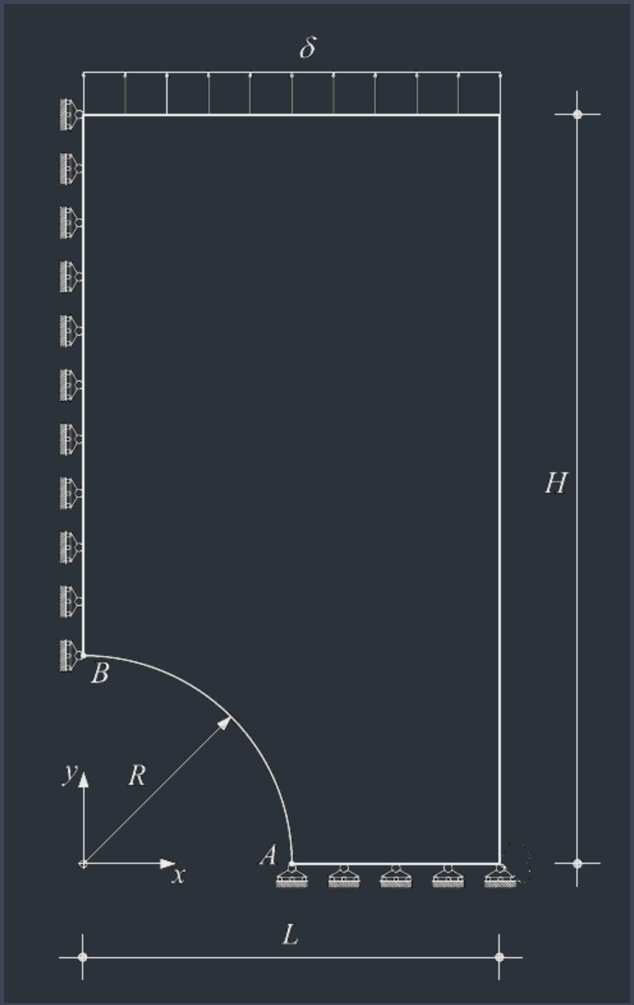


syy max. = 4.708 MPa



syy max. = 4.826 MPa

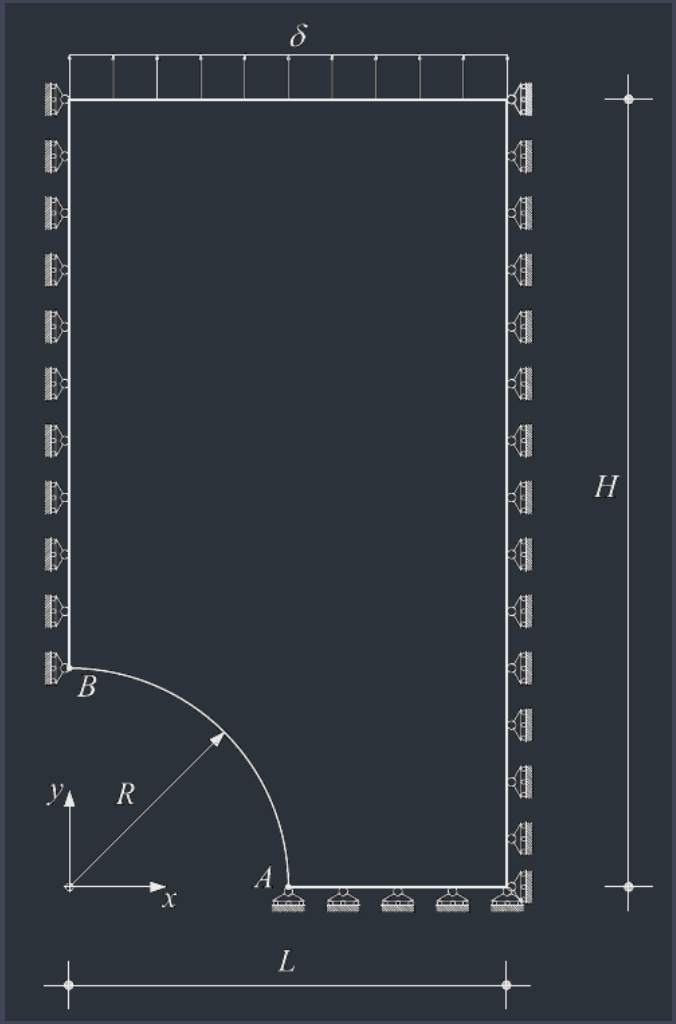
Nonlinear NIVED: Perforated Plate (Cont'd)



Displacement solution:

	NIVED (2624 dofs)	NIVED (6314 dofs)	NIVED (13458 dofs)	FEM-Q4 (49284 dofs)
u_y (B)	8.789	8.791	8.792	8.792
u_x (A)	-4.584	-4.597	-4.606	-4.609

Nonlinear NIVED: Locking Effect



Dimensions:

- H = 180 mm
- L = 100 mm
- R = 50 mm

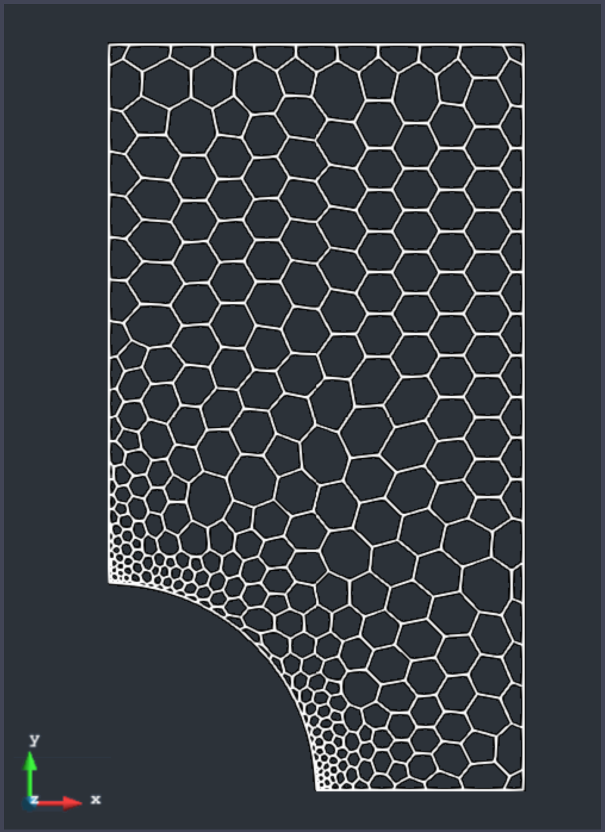
Material:

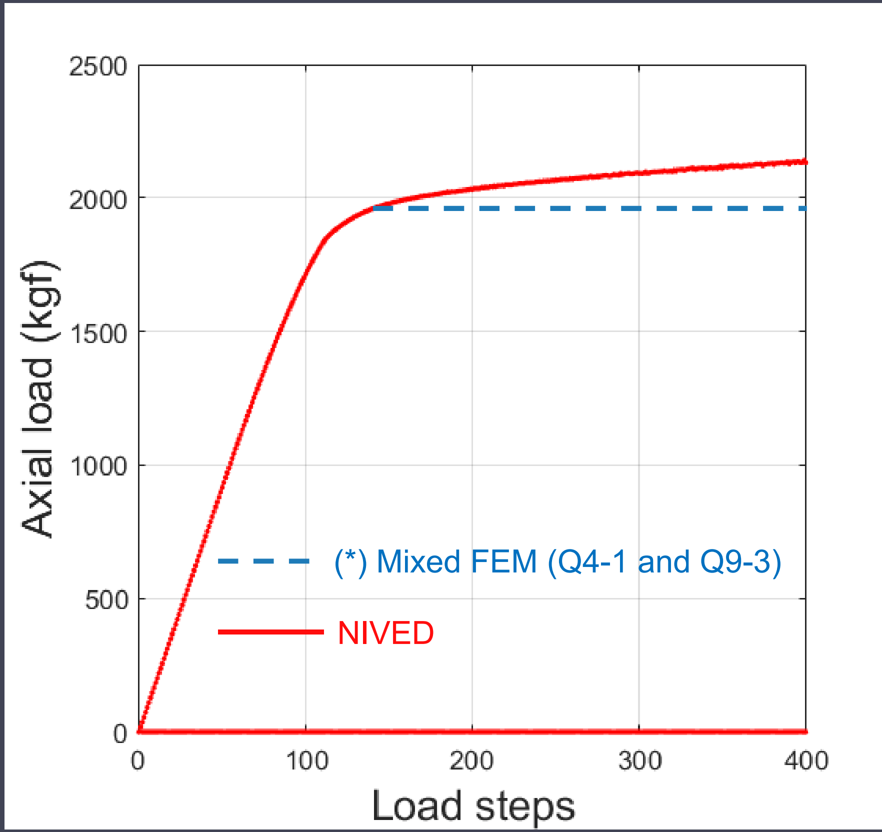
- J2-Plasticity / Plane strain
- Huber-Mises-Hencky yield function
- Perfect plasticity
- E = 7000 kgf/mm2
- ν = 0.3
- H_i = 0 MPa
- H_k = 0 MPa
- σ_{y0} = 24.3 kgf/mm2

Top displacement:

- δ = 2 mm

Nonlinear NIVED: Locking Effect (Cont'd)





(*) Zienkiewicz & Taylor, The Finite Element Method for Solid and Structural Mechanics, Elsevier Butterworth-Heinemann, Sixth Edition, 2005.

Summary and Outlook

- Developed NIVED (Nodal Integration – VED) for elastic and elastoplastic applications.
- VED provides consistency (patch test satisfaction) and optimal convergence to the nodal integration method.
- Computational cost of NIVED is similar to the computational cost of a 3-pt standard Gauss rule.
- Outlook: locking effect, 3D simulations, large deformations.

Journal Acronyms

- M3A: Mathematical Models and Methods in Applied Sciences
- CMAME: Computer Methods in Applied Mechanics and Engineering
- IJNME: International Journal for Numerical Methods in Engineering
- IJNMF: International Journal for Numerical Methods in Fluids
- CM: Computational Mechanics
- CMA: Computers & Mathematics with Applications