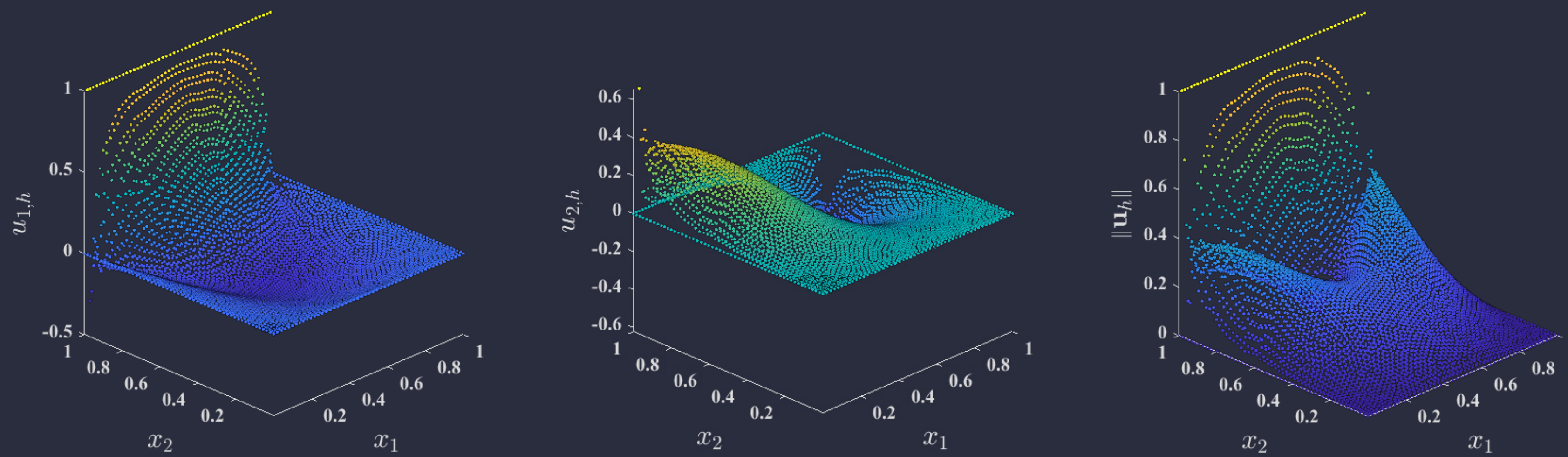
A NODE-BASED UNIFORM STRAIN VIRTUAL ELEMENT METHOD FOR ELASTIC AND INELASTIC SMALL DEFORMATION PROBLEMS



Alejandro Ortiz-Bernardin
Department of Mechanical Engineering
Universidad de Chile
www.camlab.cl/alejandro

DIMEC
INGENIERÍA MECÁNICA
UNIVERSIDAD DE CHILE

Funded through grant ANID/FONDECYT No. 1221325

CSFWGBC 2022
CSF Workshop on Generalized Barycentric Coordinates in Computer Graphics and Computational Mechanics
Ascona ● Switzerland ● June 2, 2022

Collaborators

Nodal Integration – VEM:

Rodrigo Silva-Valenzuela
Ph.D. Student
Department of Mechanical Engineering
Universidad de Chile

Remeshing, 3D:

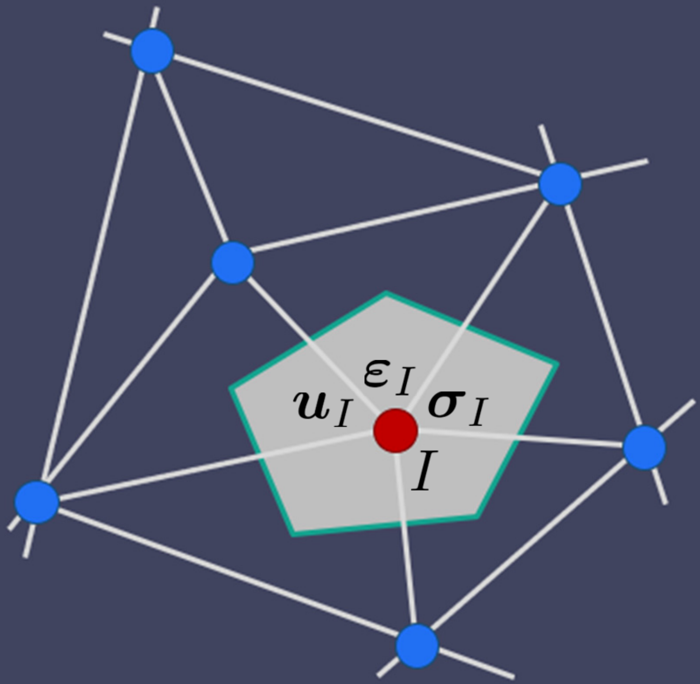
Sebastián Luza
M.Sc. Student
Department of Mechanical Engineering
Universidad de Chile

Bruno Rebolledo
M.Sc. Student
Department of Mechanical Engineering
Universidad de Chile

Mesh generation:

Sergio Salinas-Fernández
Ph.D. Student
Department of Computer Science
Universidad de Chile

Nancy Hitschfeld-Kahler
Associate Professor
Department of Computer Science
Universidad de Chile



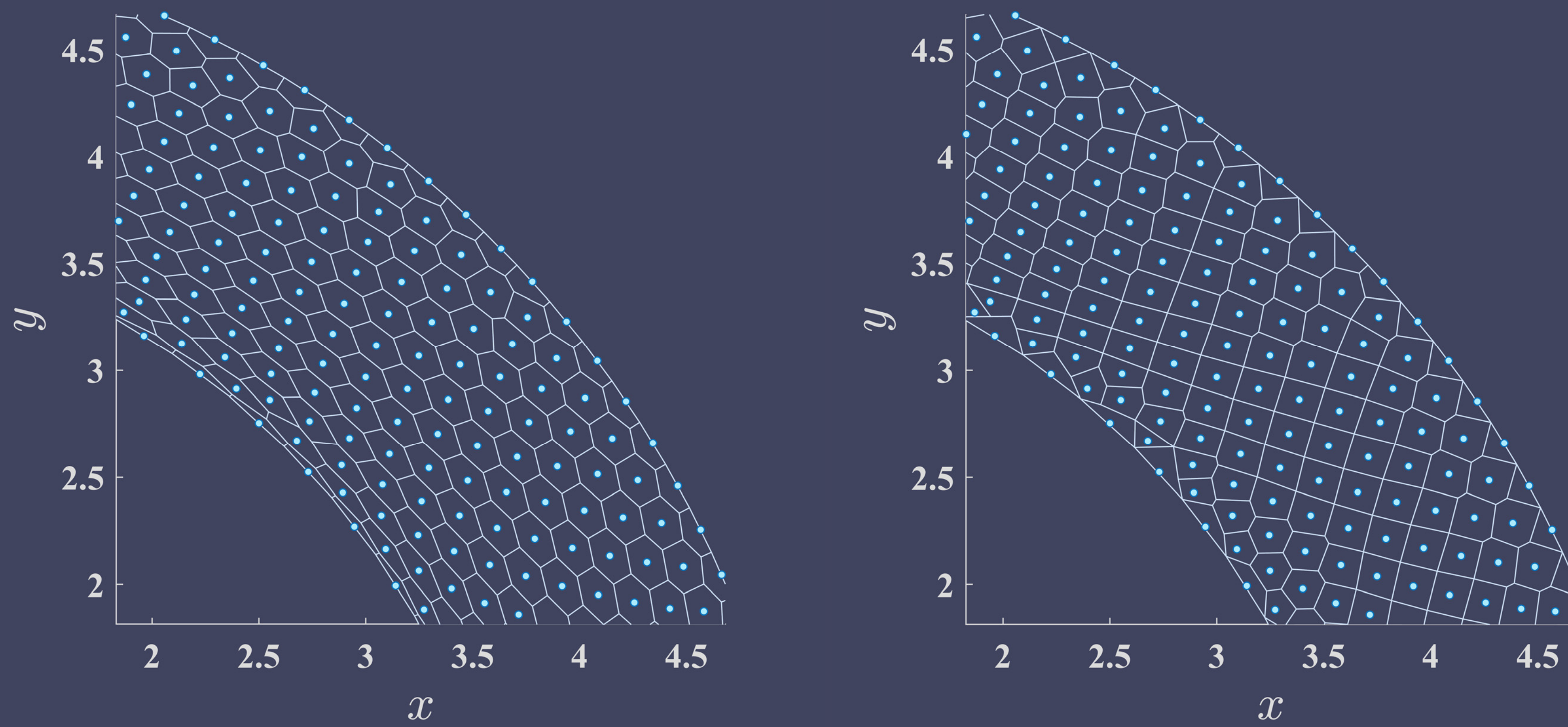
Meshfree nodal integration

What is a nodal integration scheme?

- Numerical integration performed at the nodes
- Displacements, strains and stresses are nodal quantities
- In general, internal and history-dependent variables become nodal quantities

Motivation (Cont'd)

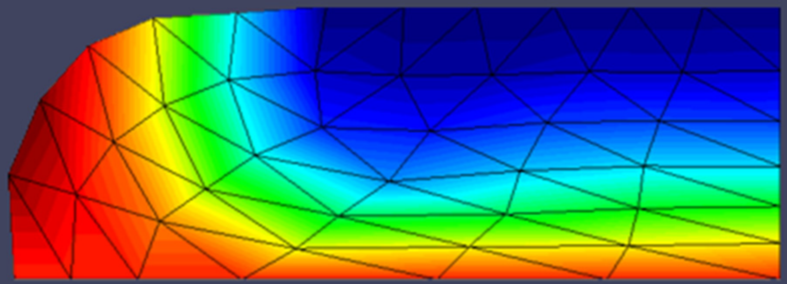
Raison d'être of nodal integration schemes



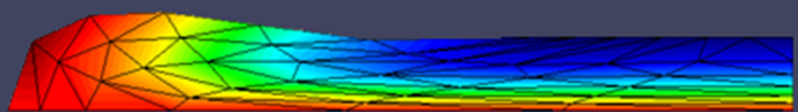
- Remeshing using the same set of nodes
- Remapping of variables is avoided

Nodal integration in computational solid mechanics

- Meshfree and finite elements [P2008]
- More popular in meshfree methods
- Meshfree is less sensitive to mesh distortions than finite elements (mesh can be deformed more before a remeshing)
- Nodal integration is cheaper than Gauss integration in meshfree methods



FEM



MESHFREE

[P2018] Puso et al. Meshfree and finite element nodal integration methods. *International Journal for Numerical Methods in Engineering* 2008; 74(3): 416–446

Motivation (Cont’d)

Nodal integration + virtual element method: Why?

- Nodal integration accelerates meshfree computations
- But it is not enough (neighbor search, optimization problem)
- Nodal integration requires stabilization

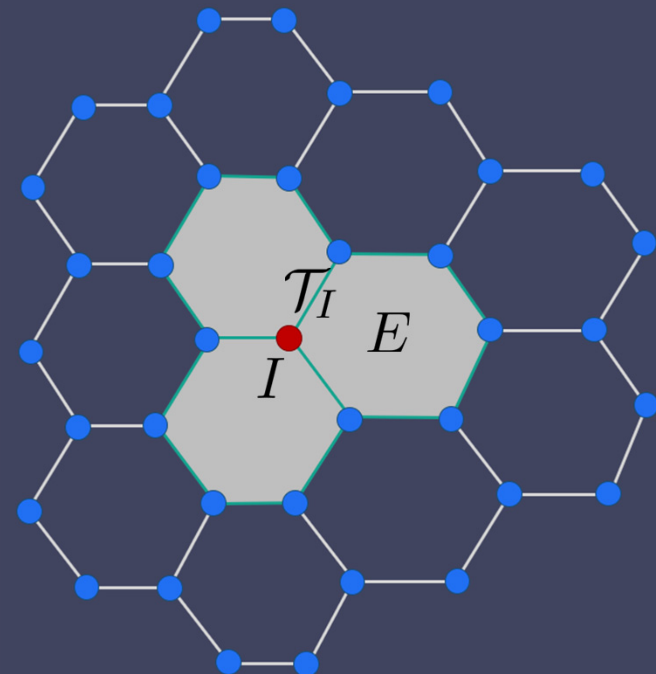
(potential contributions of the VEM to nodal integration)

- Virtual Element Method (VEM) is robust under mesh distortions
- VEM provides a stabilization procedure
- VEM is closer to FEM in terms of computational resources

Node-Based Uniform Strain

Representative nodal area:

$$|I| = \sum_{E \in \mathcal{T}_I} \frac{1}{N_E^V} |E|; \quad N_E^V = \#\text{ vertices } E$$



Strain on each virtual element:

$$\widehat{\varepsilon}(\boldsymbol{u}_h) = \frac{1}{2|E|} \int_{\partial E} (\boldsymbol{u}_h \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{u}_h) \, \mathrm{d}s$$

Node-based uniform strain:

$$\widehat{\varepsilon}_I(\boldsymbol{u}_h) = \frac{1}{|I|} \sum_{E \in \mathcal{T}_I} |E| \frac{1}{N_E^V} \widehat{\varepsilon}(\boldsymbol{u}_h) \implies$$

Nodal averaging operator:

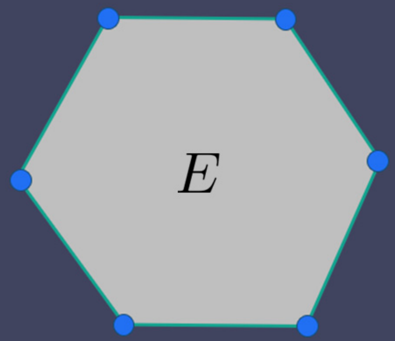
$$\boxed{\pi_I[F] = F_I = \frac{1}{|I|} \sum_{E \in \mathcal{T}_I} |E| \frac{1}{N_E^V} [F]_E}$$

[D2000] Dohrmann et al. Node-based uniform strain elements for three-node triangular and four-node tetrahedral meshes. *International Journal for Numerical Methods in Engineering* 2000; 47(9):1549–1568

Node-Based Uniform Strain VEM (NVEM)

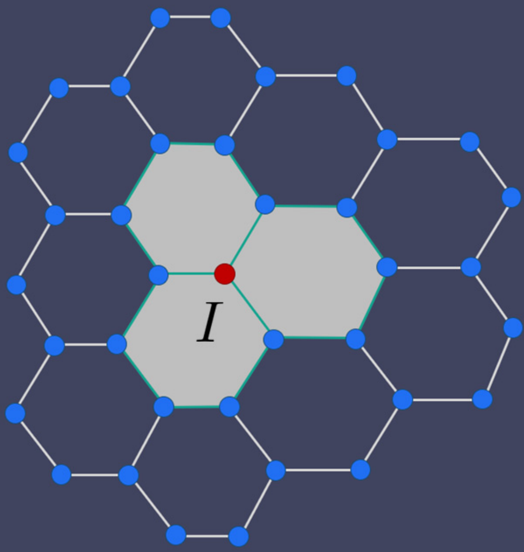
VEM bilinear form at element level:

$$
\begin{aligned}
a_{h,E}(\boldsymbol{u}_h, \boldsymbol{v}_h) &= a_E(\Pi \boldsymbol{u}_h, \Pi \boldsymbol{v}_h) \\
&\quad + s_E(\boldsymbol{u}_h - \Pi \boldsymbol{u}_h, \boldsymbol{v}_h - \Pi \boldsymbol{v}_h)
\end{aligned}
$$



NVEM bilinear form at node level:

$$
\begin{aligned}
a_{h,I}(\boldsymbol{u}_h, \boldsymbol{v}_h) &= a_I(\pi_I[\Pi \boldsymbol{u}_h], \pi_I[\Pi \boldsymbol{v}_h]) \\
&\quad + s_I(\pi_I[\boldsymbol{u}_h - \Pi \boldsymbol{u}_h], \pi_I[\boldsymbol{v}_h - \Pi \boldsymbol{v}_h])
\end{aligned}
$$



[OB2022] Ortiz-Bernardin et al. Node-based uniform strain virtual elements for compressible and nearly incompressible plane elasticity. arXiv:2204.13825v2 [math.NA]

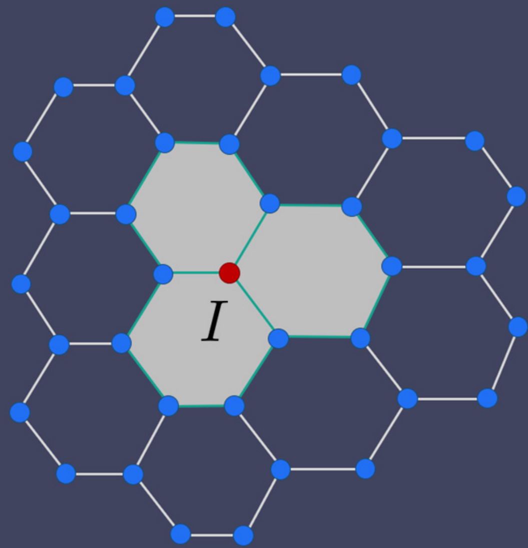
Node-Based Uniform Strain VEM (NVEM) (Cont'd)

NVEM bilinear form at node level:

$$
\boxed{\begin{aligned} a_{h,I}(\boldsymbol{u}_h,\boldsymbol{v}_h) = {} & |I|\,\widehat{\boldsymbol{\varepsilon}}_I^{\mathsf{T}}(\boldsymbol{v}_h)\,\boldsymbol{D}\,\widehat{\boldsymbol{\varepsilon}}_I(\boldsymbol{u}_h) \\ & + (\boldsymbol{1}-\mathit{\Pi})_I^{\mathsf{T}} s_I(\boldsymbol{v}_h,\boldsymbol{u}_h)(\boldsymbol{1}-\mathit{\Pi})_I \end{aligned}}
$$

$$
\widehat{\boldsymbol{\varepsilon}}_I = \pi_I[\,\widehat{\boldsymbol{\varepsilon}}\,]; \quad (\boldsymbol{1}-\mathit{\Pi})_I = \pi_I[\,\boldsymbol{1}-\mathit{\Pi}\,]; \quad s_I = \pi_I[\,s_E\,]
$$

$$
\pi_I[\,F\,] = \tfrac{1}{|I|}\textstyle\sum_{E\in\mathcal{T}_I}|E|\tfrac{1}{N_E^V}[\,F\,]_E
$$



NVEM bilinear form at global level:

$$
a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = \sum_{I=1}^{N}\left[|I|\,\widehat{\boldsymbol{\varepsilon}}_I^{\mathsf{T}}(\boldsymbol{v}_h)\,\boldsymbol{D}\,\widehat{\boldsymbol{\varepsilon}}_I(\boldsymbol{u}_h) + (\boldsymbol{1}-\mathit{\Pi})_I^{\mathsf{T}} s_I(\boldsymbol{v}_h,\boldsymbol{u}_h)(\boldsymbol{1}-\mathit{\Pi})_I\right]
$$

[OB2022] Ortiz-Bernardin et al. Node-based uniform strain virtual elements for compressible and nearly incompressible plane elasticity. arXiv:2204.13825v2 [math.NA]

Stabilization in the NVEM

Stabilization can render the solution too ‘stiff’ in the nearly incompressible limit. To mitigate this, a modified constitutive matrix is used in the stability part [P2018].

$$\boxed{(\boldsymbol{s}_I)_{i,i} = \max\left(\left(\boldsymbol{v}_h^{\mathsf{T}}(\boldsymbol{x}_I)\,\boldsymbol{u}_h(\boldsymbol{x}_I)\right)_{i,i},\left(|I|\,\widehat{\boldsymbol{\varepsilon}}_I^{\mathsf{T}}(\boldsymbol{v}_h)\,\widetilde{\boldsymbol{D}}\,\widehat{\boldsymbol{\varepsilon}}_I(\boldsymbol{u}_h)\right)_{i,i}\right),}$$

$$\widetilde{\boldsymbol{D}} = \boldsymbol{D}\left(\widetilde{E},\widetilde{\nu}\right), \quad \widetilde{E} = \frac{\widetilde{\nu}\left(3\widetilde{\lambda}+2\widetilde{\mu}\right)}{\widetilde{\lambda}+\widetilde{\mu}}, \quad \widetilde{\nu} = \frac{\widetilde{\lambda}}{2\left(\widetilde{\lambda}+\widetilde{\mu}\right)},$$

$$\widetilde{\mu} := \mu, \quad \widetilde{\lambda} := \min\left(\lambda, 25\widetilde{\mu}\right)$$

[P2018] Puso et al. Meshfree and finite element nodal integration methods. *International Journal for Numerical Methods in Engineering* 2008; 74(3): 416–446

Stabilization in the NVEM (Cont'd)

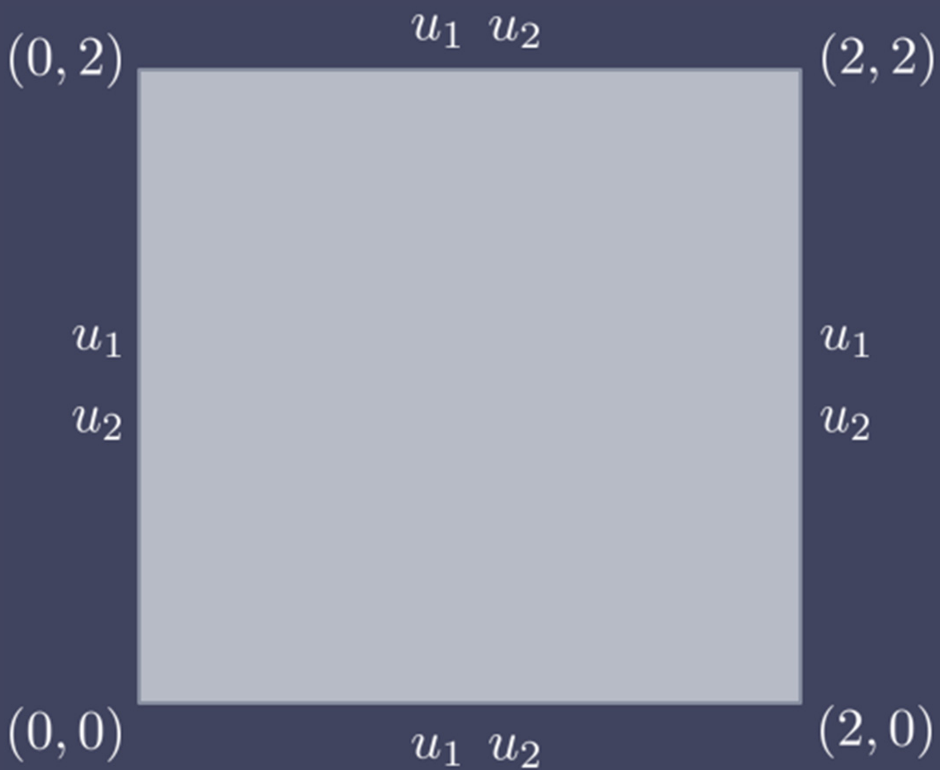
An alternative stabilization that also mitigates the 'stiff' behavior of the full stabilization is [OB2022]:

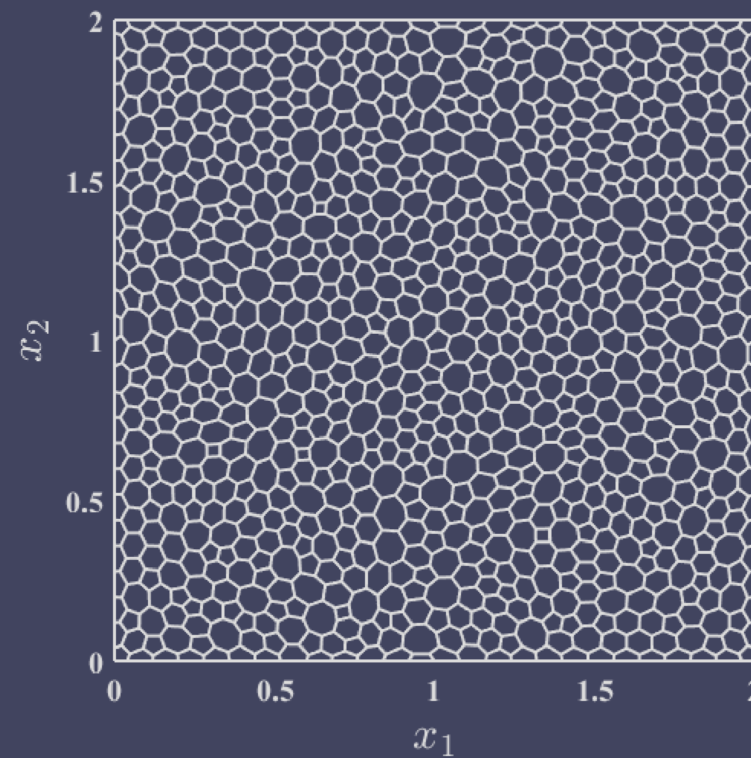
$$\boxed{(\boldsymbol{s}_I)_{i,i} = \max\left(\left(\boldsymbol{v}_h^\intercal(\boldsymbol{x}_I)\,\boldsymbol{u}_h(\boldsymbol{x}_I)\right)_{i,i}, \left(|I|\,\widehat{\boldsymbol{\varepsilon}}_I^\intercal(\boldsymbol{v}_h)\,\boldsymbol{D}_\mu\,\widehat{\boldsymbol{\varepsilon}}_I(\boldsymbol{u}_h)\right)_{i,i}\right),}$$

$$\boldsymbol{D}_\mu = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \quad \text{or} \quad \boldsymbol{D}_\mu = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

[OB2022] Ortiz-Bernardin et al. Node-based uniform strain virtual elements for compressible and nearly incompressible plane elasticity. arXiv:2204.13825v2 [math.NA]

Numerical Tests: Colliding Flow

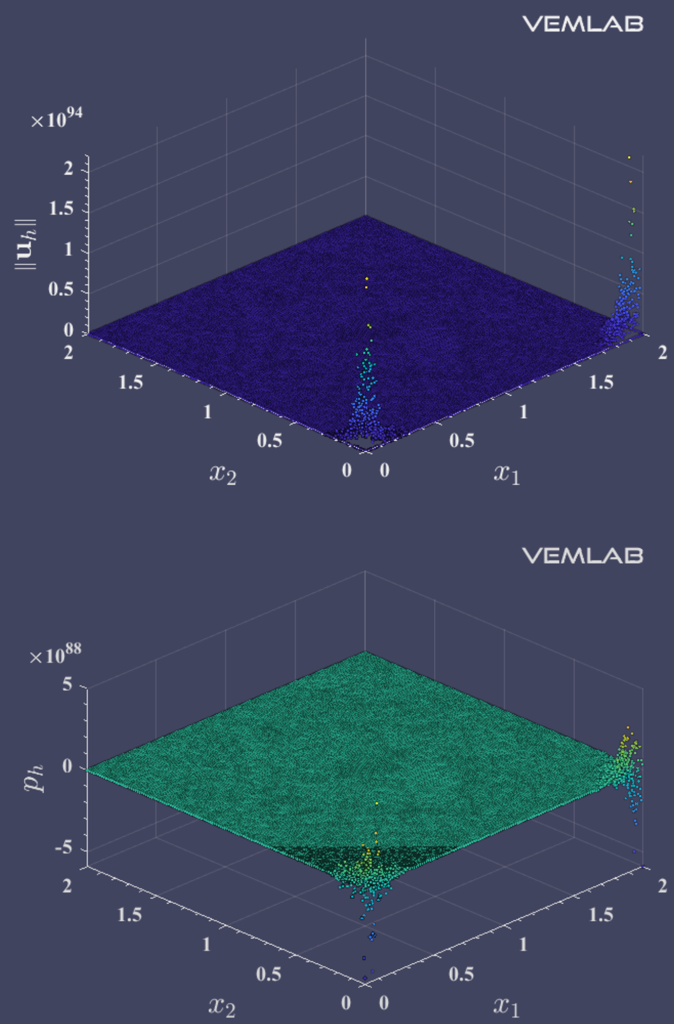




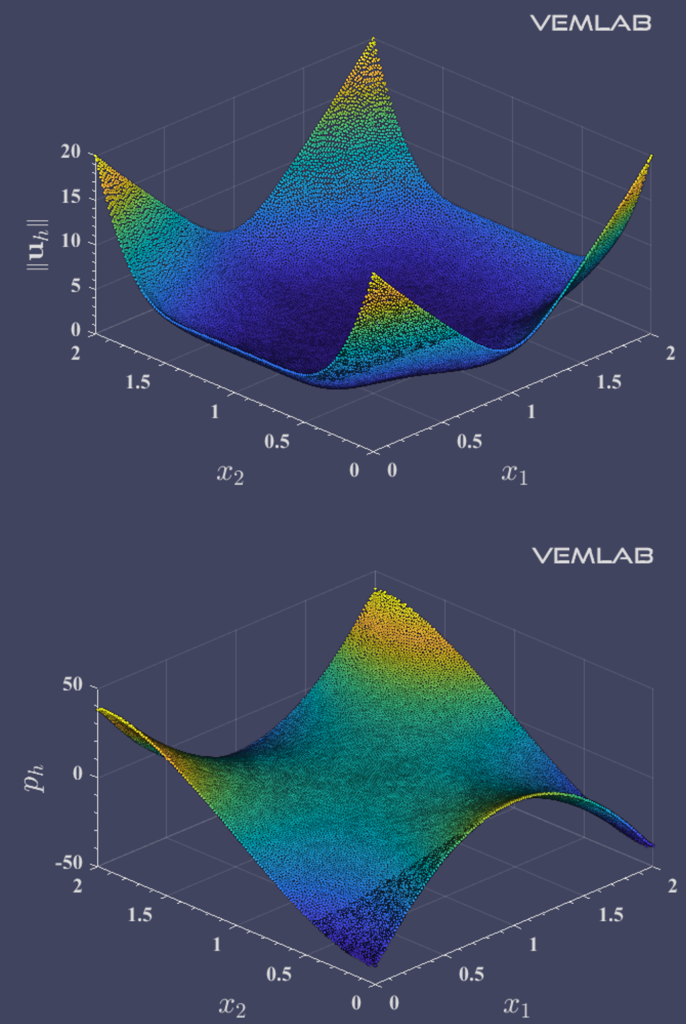
$$u_1 = 20(x_1 - 1)(x_2 - 1)^3$$
$$u_2 = 5(x_1 - 1)^4 - 5(x_2 - 1)^4$$

$$\begin{aligned} \lambda &= 5 \times 10^7 \\ \mu &= 1 \end{aligned} \implies \nu = 0.49999999$$

Numerical Tests: Colliding Flow (Cont’d)

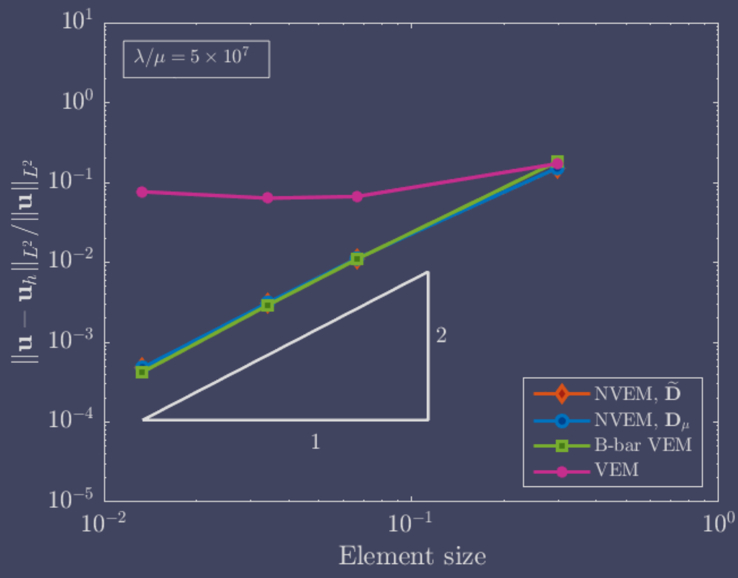


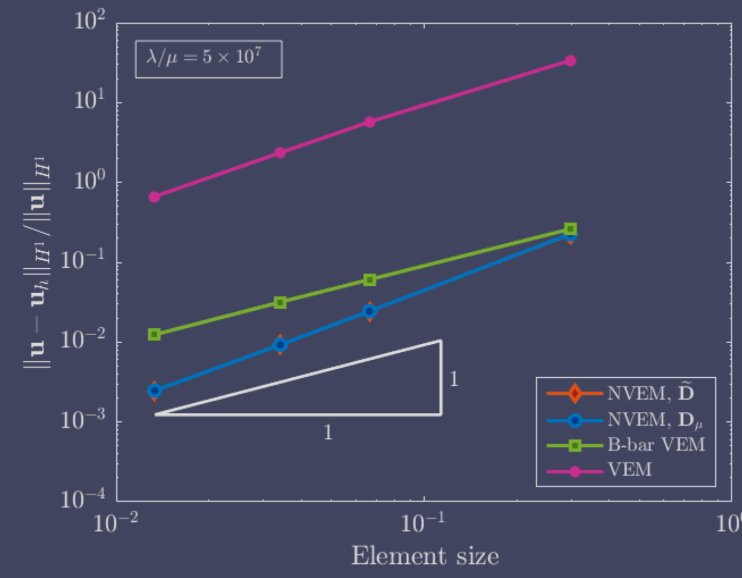
(w/o stability)

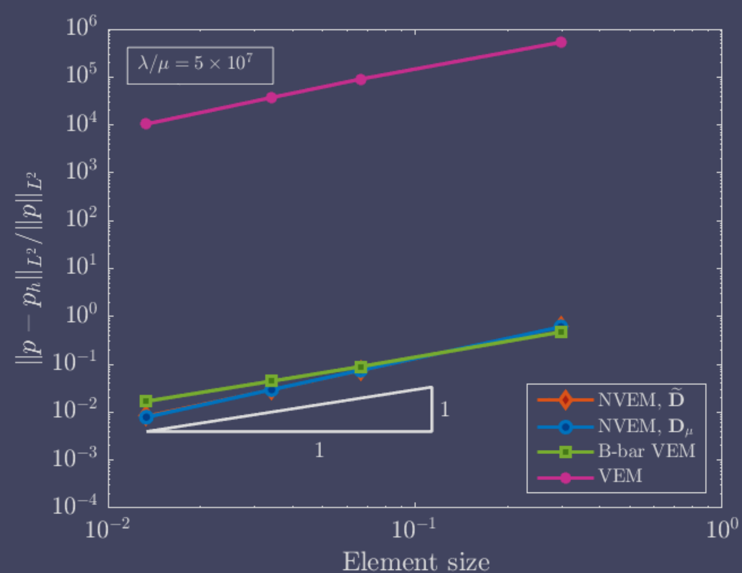


($\boldsymbol{D}_\mu$ stability)

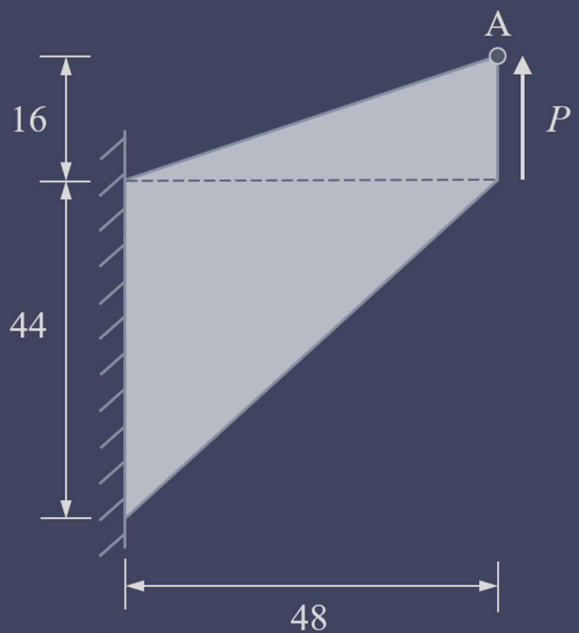
Numerical Tests: Colliding Flow (Cont'd)

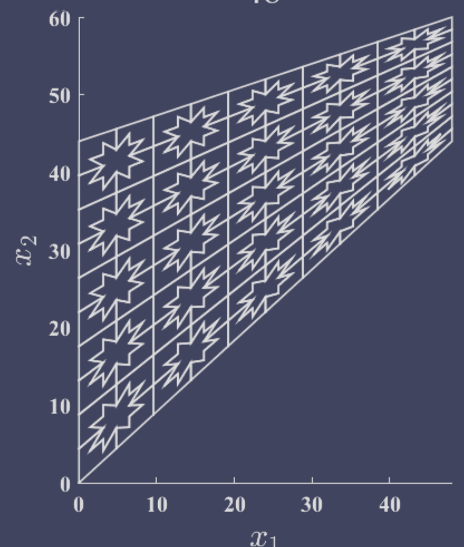




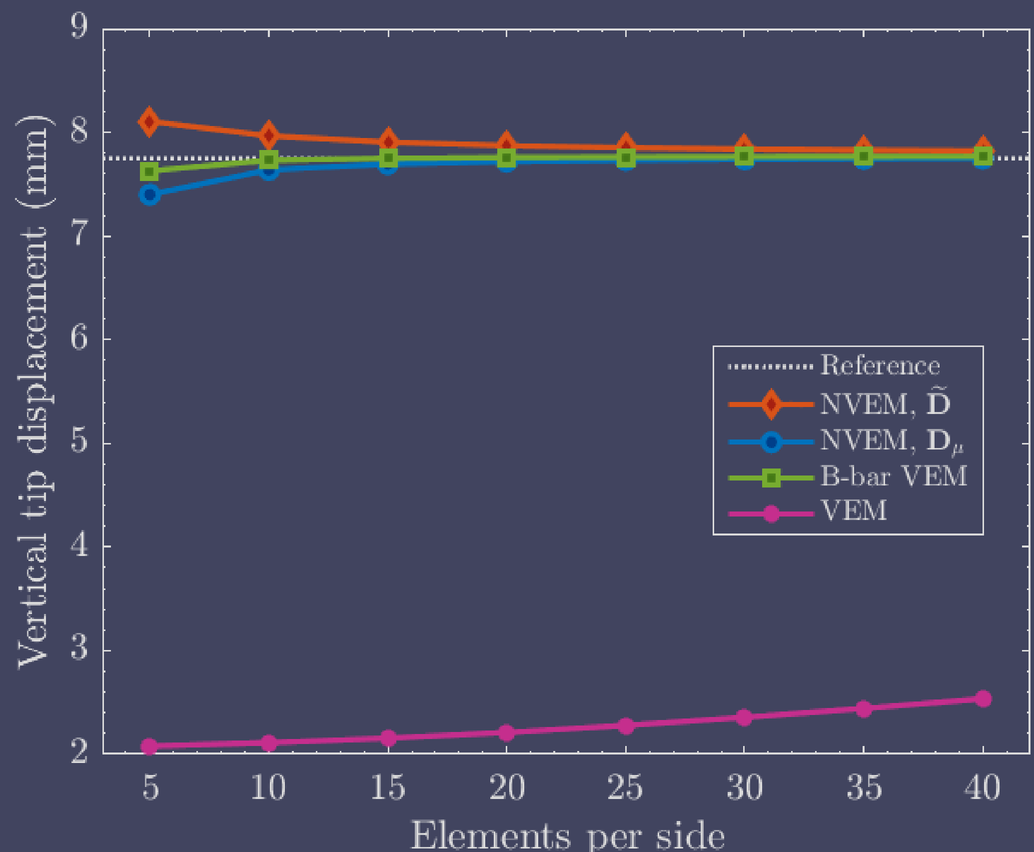


Numerical Tests: Cook's Membrane

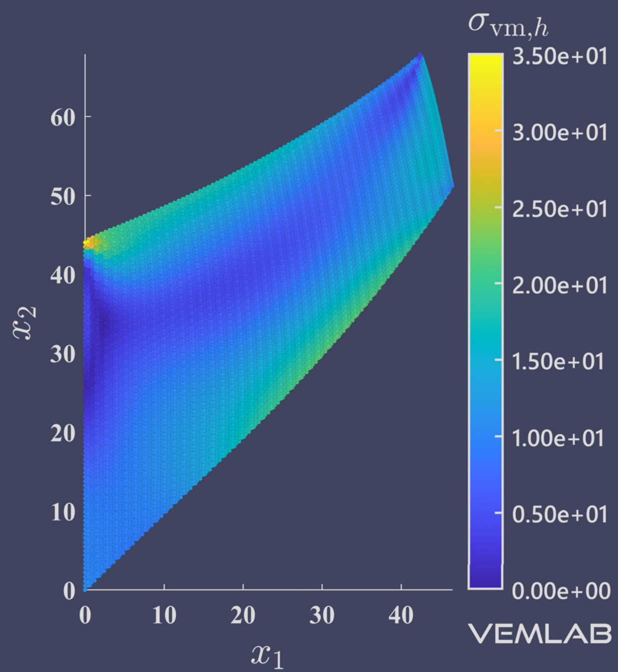




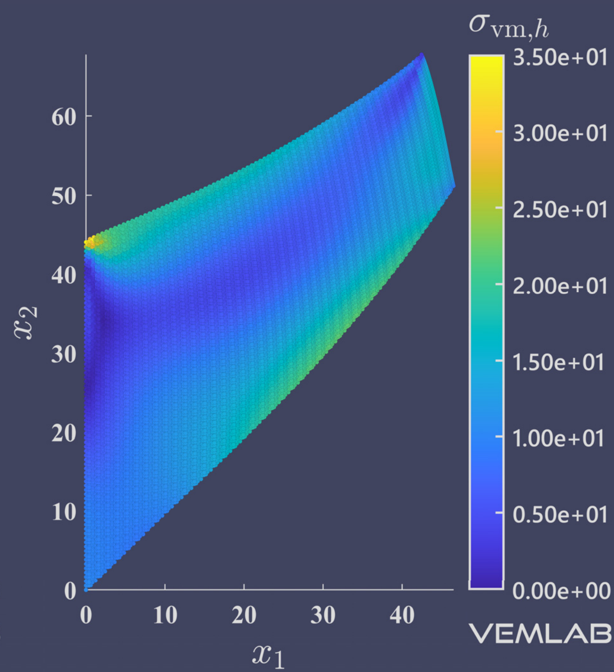
$$P = 100\ \mathrm{N},\ E_{\mathrm{Y}} = 250\ \mathrm{MPa},\ \nu = 0.4999$$



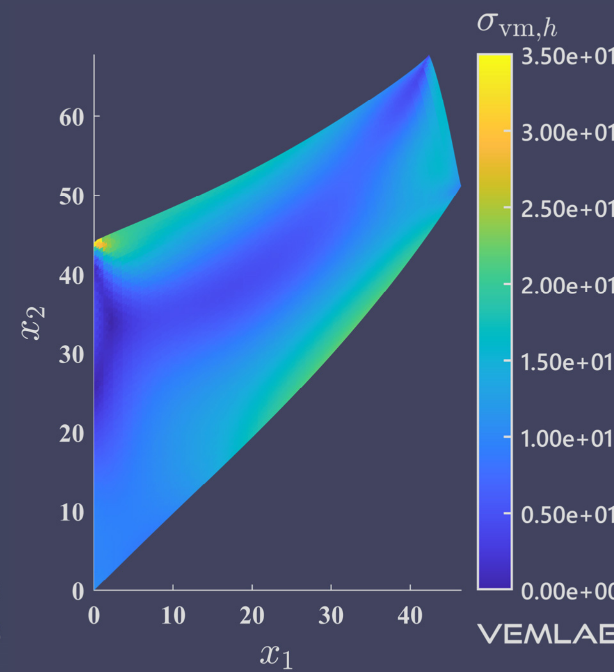
Numerical Tests: Cook's Membrane (Cont'd)



($\widetilde{\boldsymbol{D}}$ stability)

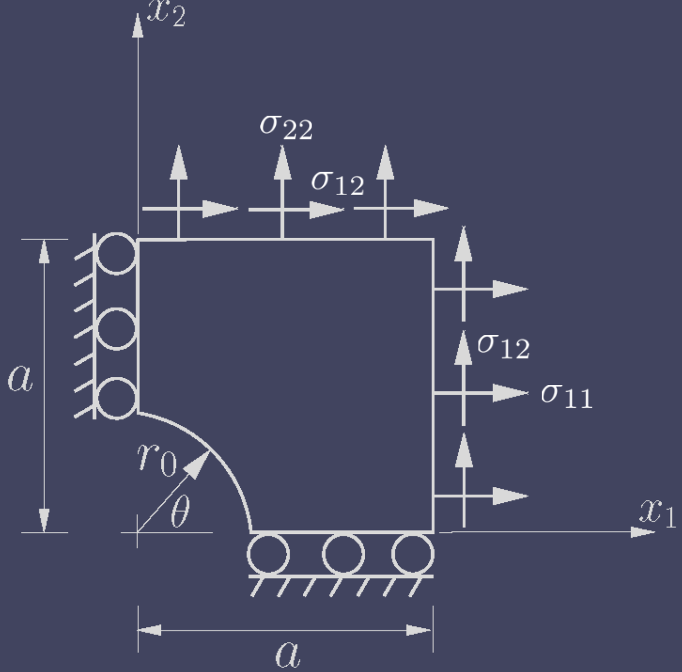


($\boldsymbol{D}_\mu$ stability)



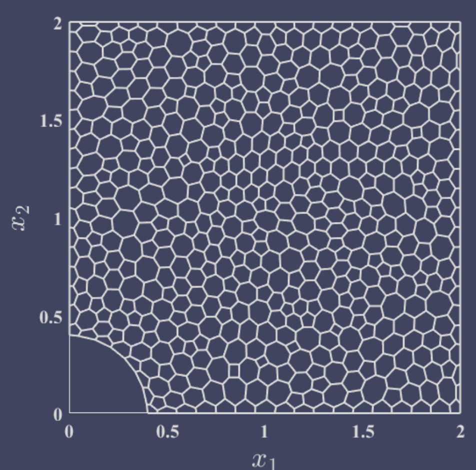
(B-bar VEM)

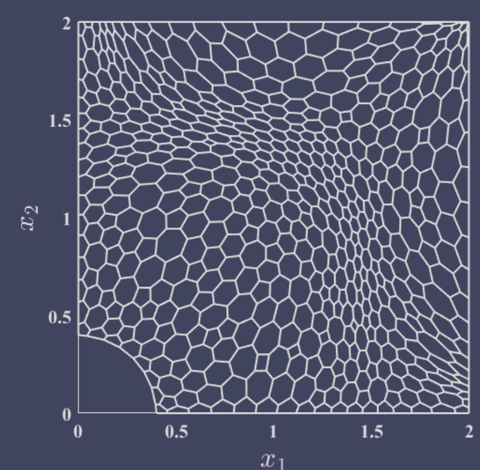
Numerical Tests: Plate With Circular Hole

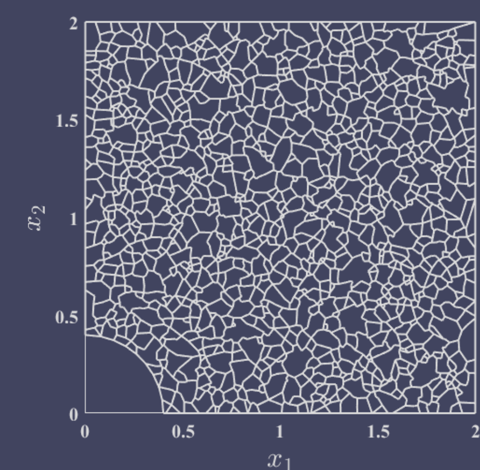


$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} T\left(1 - \frac{r_0^2}{r^2}\left(\frac{3}{2}\cos 2\theta + \cos 4\theta\right) + \frac{3r_0^4}{2r^4}\cos 4\theta\right) \\ -T\left(\frac{r_0^2}{r^2}\left(\frac{1}{2}\cos 2\theta - \cos 4\theta\right) + \frac{3r_0^4}{2r^4}\cos 4\theta\right) \\ -T\left(\frac{r_0^2}{r^2}\left(\frac{1}{2}\sin 2\theta + \sin 4\theta\right) - \frac{3r_0^4}{2r^4}\sin 4\theta\right) \end{bmatrix}
$$

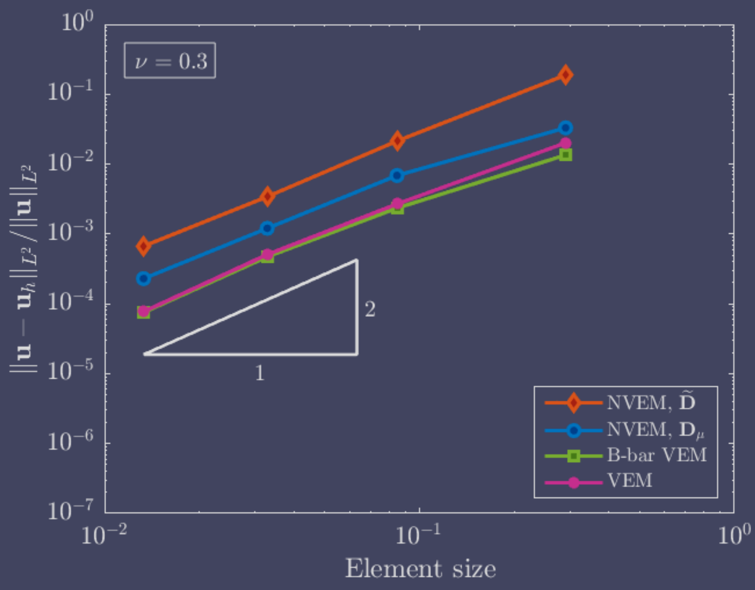
$$T = 100,\ r_0 = 1,\ a = 5$$

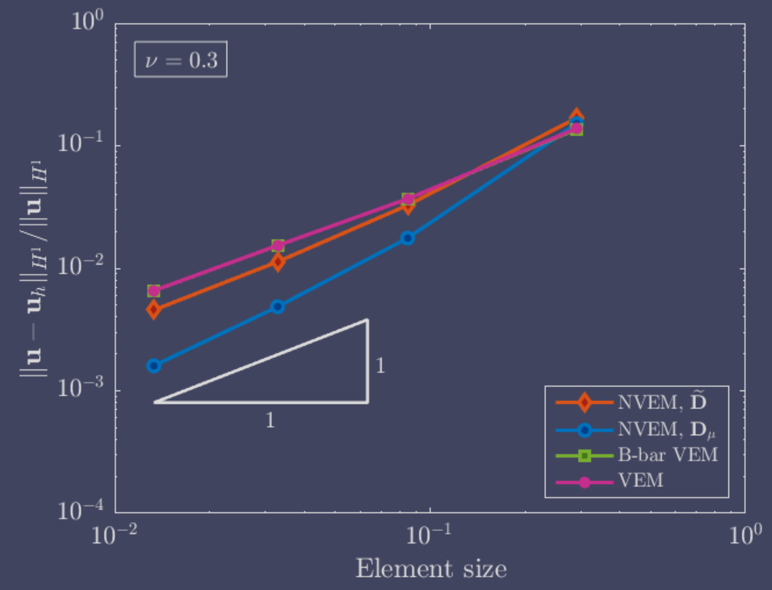


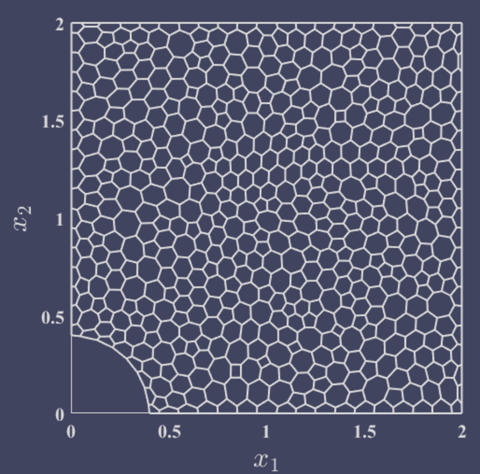




Numerical Tests: Plate With Circular Hole (Cont'd)



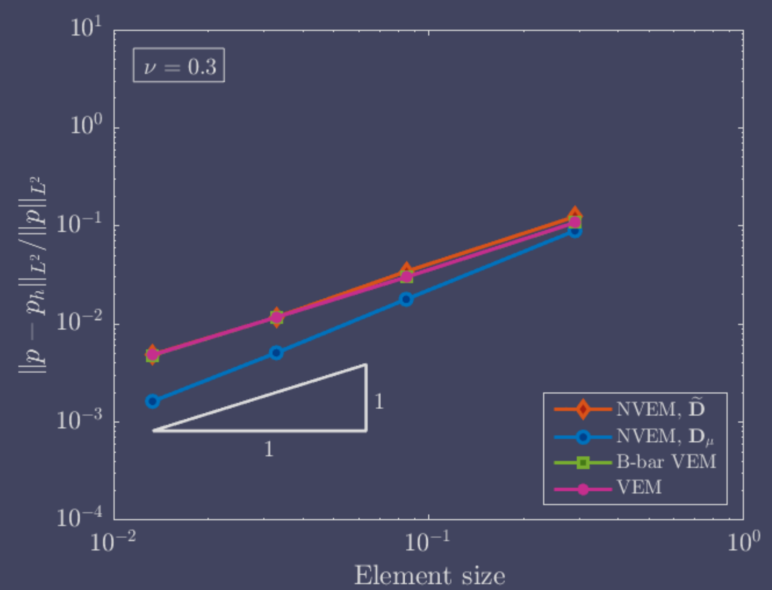




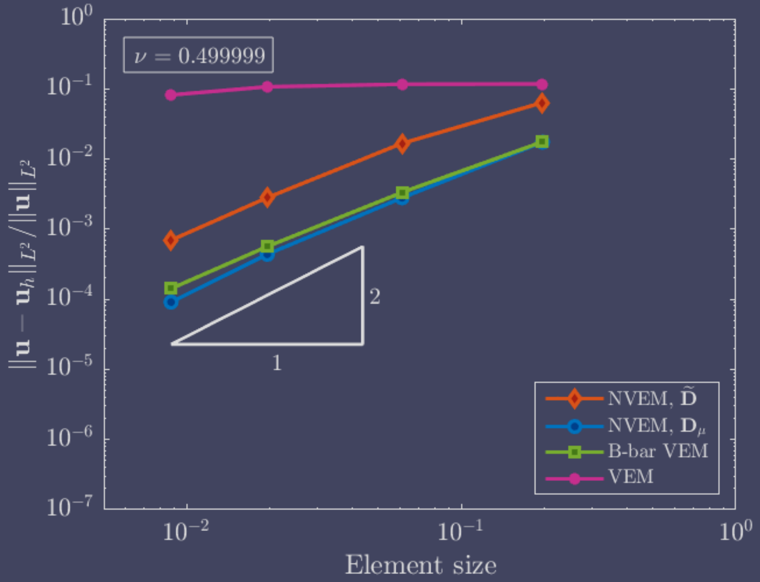
Plane stress

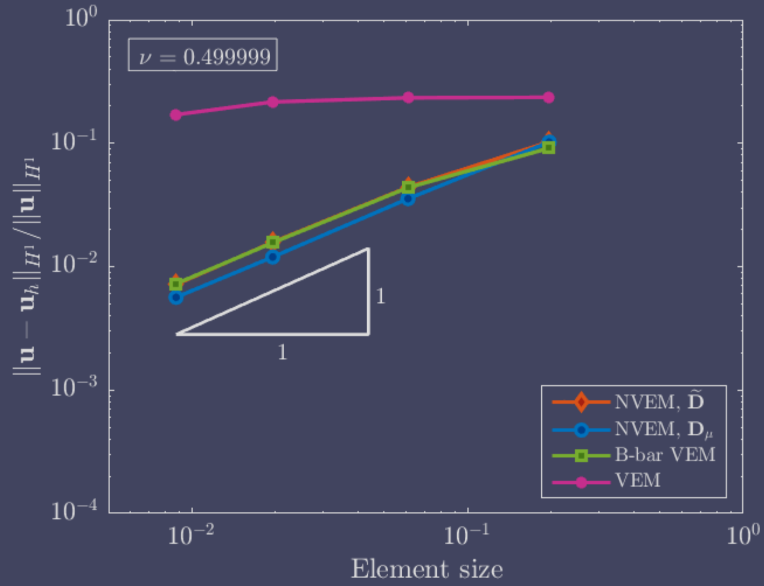
$E_{\mathrm{Y}} = 10^3$

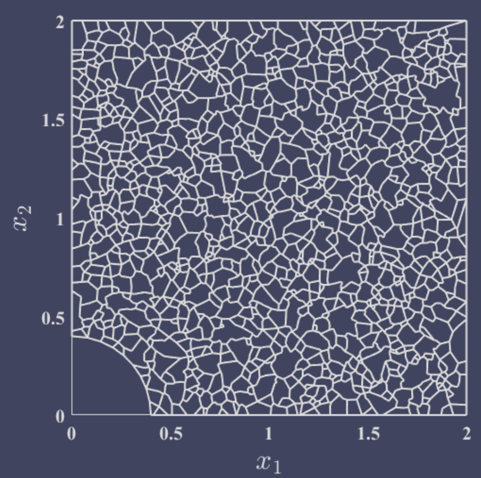
$\nu = 0.3$



Numerical Tests: Plate With Circular Hole (Cont'd)



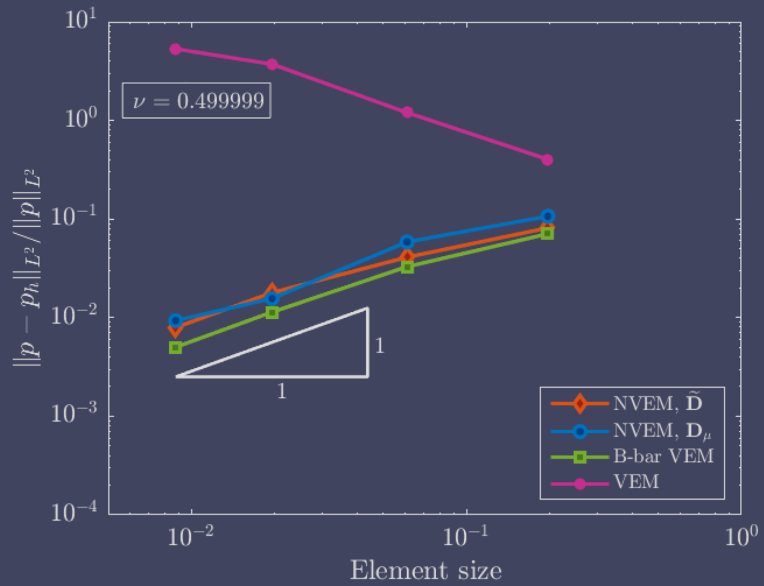




Plane strain

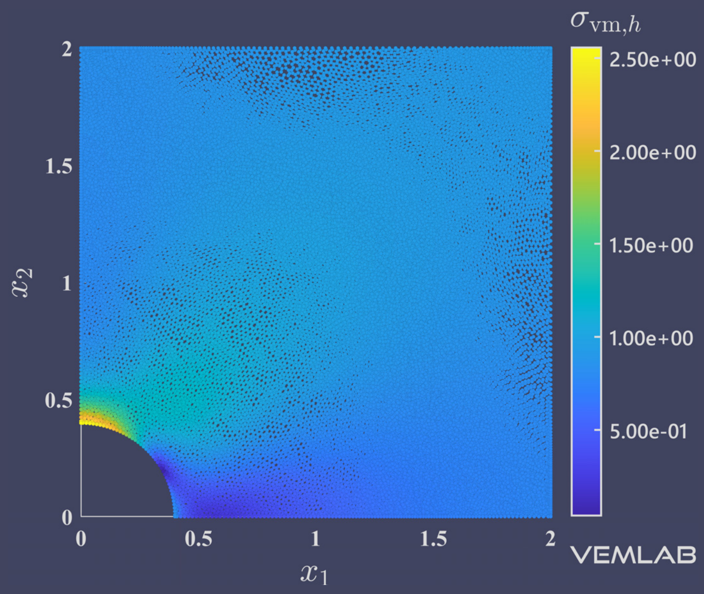
$$E_{\mathrm{Y}} = 10^3$$

$$\nu = 0.499999$$

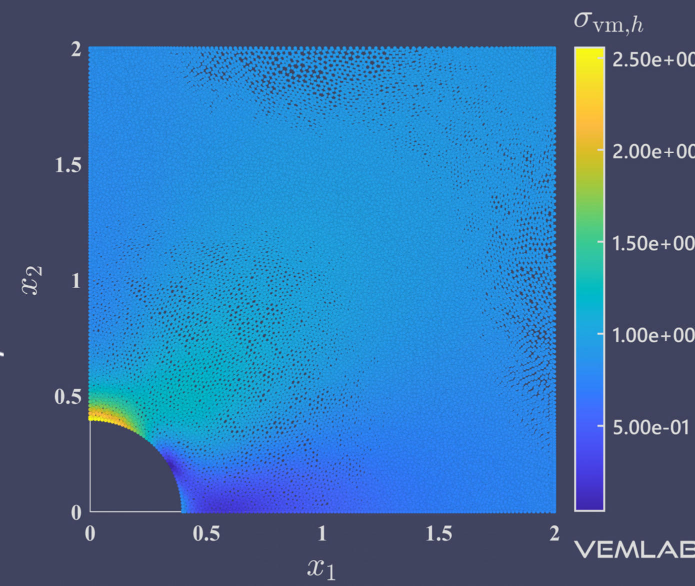


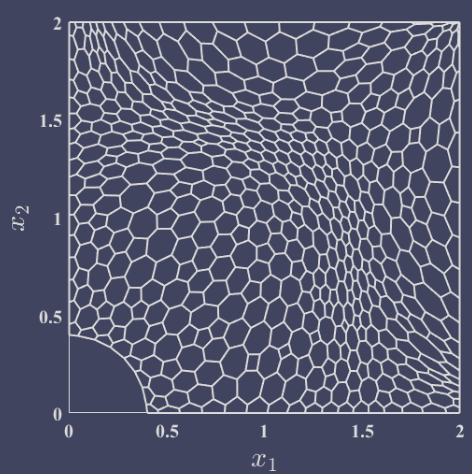
Numerical Tests: Plate With Circular Hole (Cont'd)

($\widetilde{\boldsymbol{D}}$ stability)



($\boldsymbol{D}_\mu$ stability)



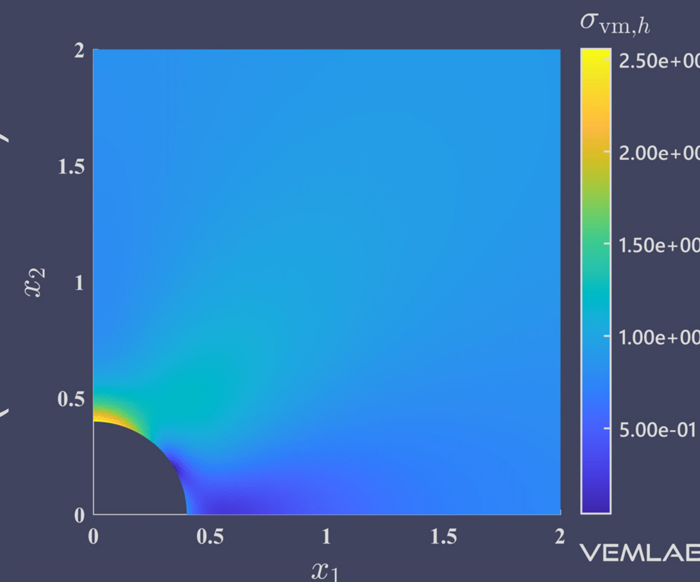


Plane strain

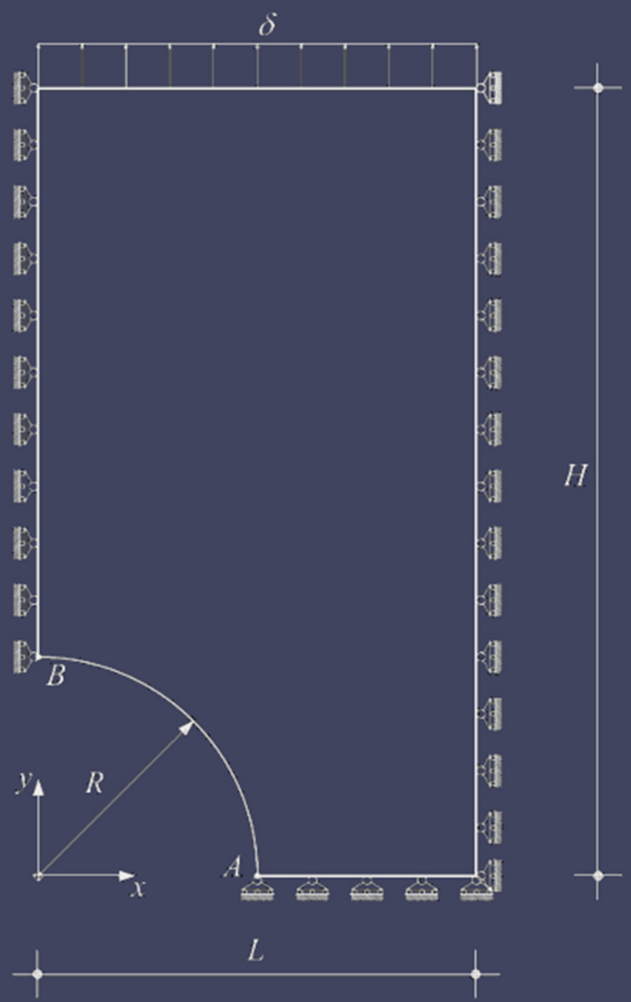
$$E_{\mathrm{Y}} = 10^3$$

$$\nu = 0.499999$$

(B-bar VEM)



Numerical Tests: Elastoplastic Perforated Plate



Dimensions:

- H = 180 mm
- L = 100 mm
- R = 50 mm

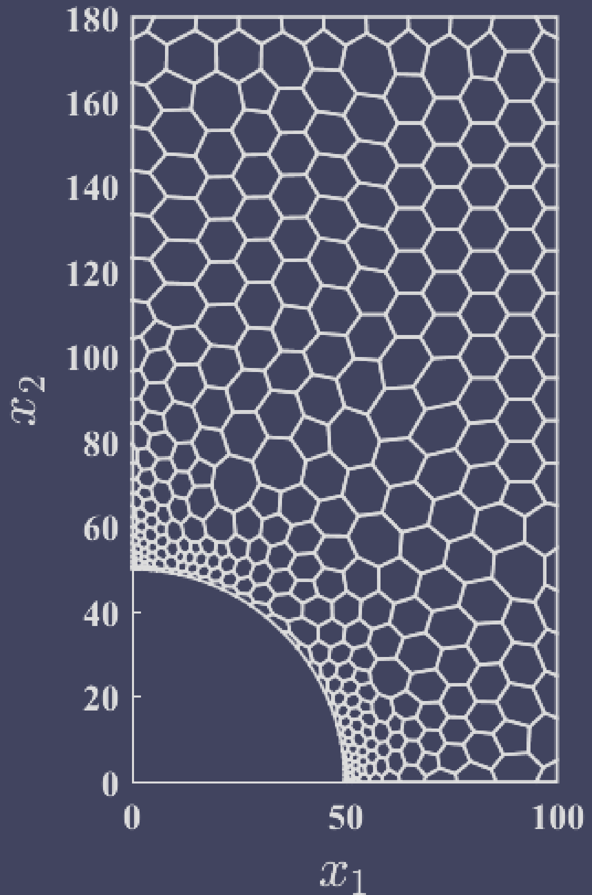
Material:

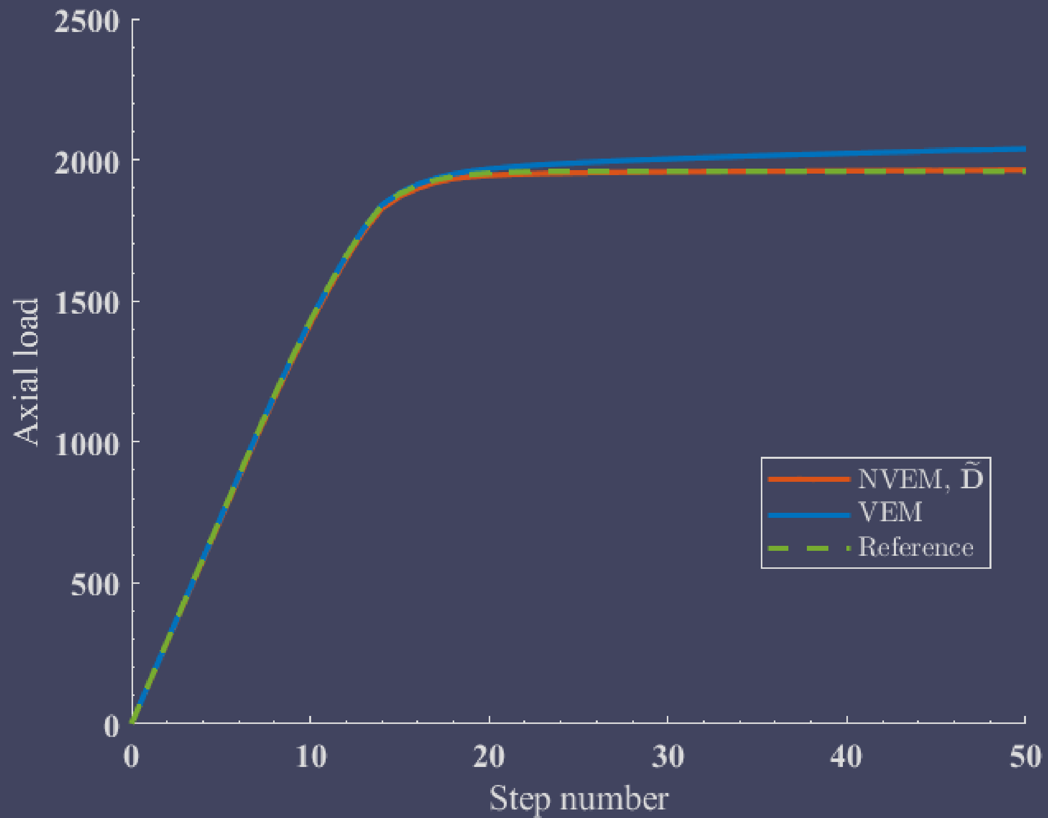
- J2-Plasticity / Plane strain
- Perfect plasticity
- E = 7000 kgf/mm2
- ν = 0.3
- σ_{y0} = 24.3 kgf/mm2

Top displacement:

- δ = 2 mm

Num. Tests: Elastoplastic Perforated Plate (Cont'd)

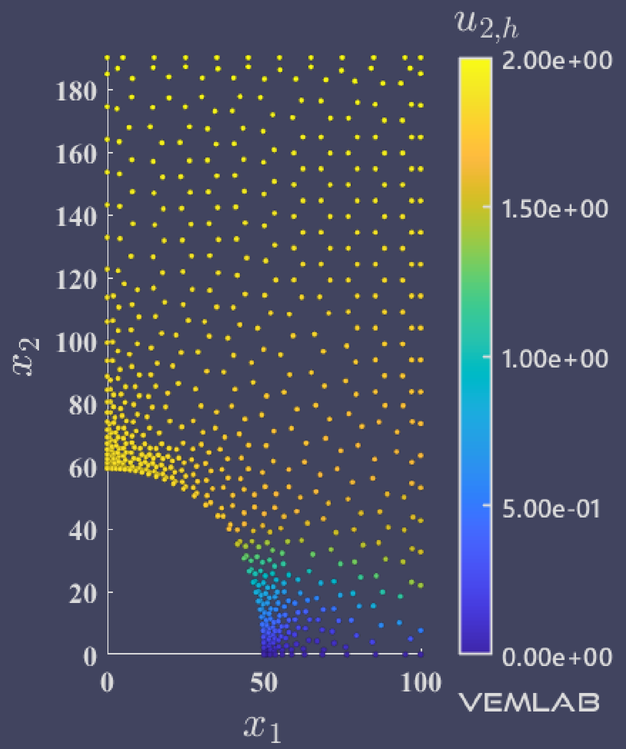




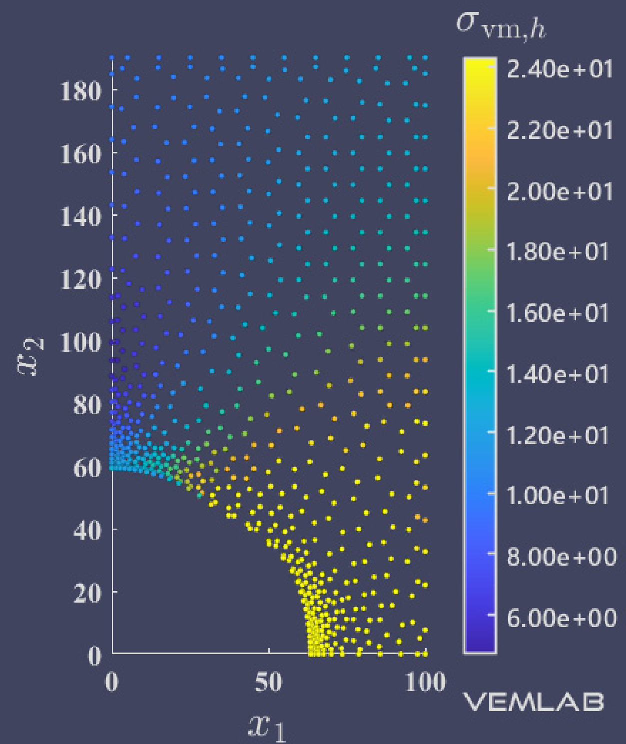
Reference solution: Q9/3 – Mixed [Z2014]

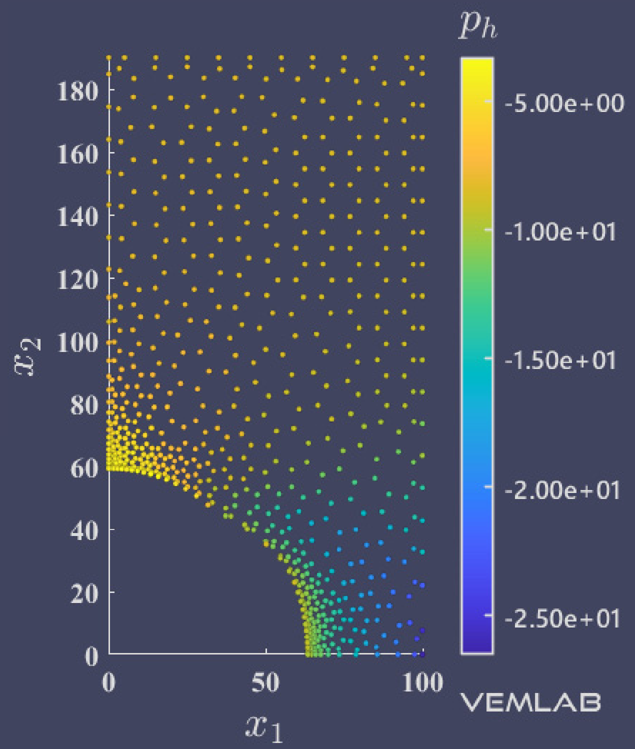
[Z2014] Zienkiewicz et al., The Finite Element Method for Solid and Structural Mechanics, Elsevier Butterworth-Heinemann, Seventh Edition, 2014.

Num. Tests: Elastoplastic Perforated Plate (Cont'd)



NIVEM - $\widetilde{D}$ solutions





Summary and Outlook

- Developed NVEM (Nodal Integration – VEM) for elastic and elastoplastic applications
- Several examples attested that NVEM is optimally convergent
- NVEM is devoid of volumetric locking in nearly incompressible elastic and elastic perfectly plastic materials
- Outlook: 3D simulations, large deformations with remeshing

Resources

Theory:

- A node-based uniform strain virtual element method for compressible and nearly incompressible elasticity (https://arxiv.org/abs/2204.13825)

VEM Solver:

- VEMLAB: A MATLAB Library for the Virtual Element Method (https://camlab.cl/software/vemlab/)

Polygonal Mesh Generation:

- POLYLLA: polygonal meshing algorithm based on terminal-edge regions (https://doi.org/10.1007/s00366-022-01643-4)
- Delynoi: an object-oriented C++ library for the generation of polygonal meshes (https://camlab.cl/software/delynoi/)